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A sub-Riemannian Gauss-Bonnet theorem for surfaces in contact manifolds

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Abstract
We obtain a sub-Riemannian version of the classical Gauss-Bonnet theorem. We consider subsurfaces of three-dimensional contact sub-Riemannian manifolds. Using a family of taming Riemannian metrics on the surface which approach a sub-Riemannian metric, we carefully study the usual Gauss-Bonnet formula under this limit. We are then able to recover the Euler characteristic of the surface from the geometry around the surface's characteristic set, i.e., the points where the tangent space to the surface and contact structure coincide. For the case of surfaces with boundary, we also give a Gauss-Bonnet type result which includes limits related to the boundary's curvature.

Keywords Gauss Bonnet theorem · Contact manifolds · Sub-Riemannian geometry · Characteristic foliation

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1 Introduction

The classical Gauss-Bonnet theorem shows that it is possible to recover purely topological information of a surface from the choice of a smooth structure and a Riemannian

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metric. In this paper, we show that such topological information can also be obtained from the restriction of a sub-Riemannian geometric structure, even though the induced metric on the surface will not induce its topology.

Consider a three-dimensional connected manifold M with a contact distribution E. A smoothly varying inner product g_E defined only on E is called a sub-Riemannian metric. This geometric structure induces a distance d_{g_E} on M, which, although not Lipschitz equivalent to any Riemannian distance, will induce the manifold topology on M [10, 18]. Given g_E and an orientation of E, there is a unique choice of a vector field Z transverse to E called the Reeb vector field. We can then extend the sub-Riemannian metric to a Riemannian metric $g^\varepsilon = \langle \cdot, \cdot \rangle_\varepsilon$ on M by making Z orthogonal to E and $\|Z\|_\varepsilon = 1/\sqrt{\varepsilon}$. Thus, the length of all vectors outside E goes to infinity as $\varepsilon \to 0$. The Riemannian distance d_{g^ε} converges to the sub-Riemannian distance d_{g_E}, and this convergence is uniform on compact sets [8].

When we restrict to a subsurface $\Sigma \subseteq M$, the situation is quite different. If h^ε is the induced Riemannian metric on Σ from g^ε, then d_{h^ε} does not converge to a metric compatible with the topology if it converges at all; see [4, 6] for details. Seeing that this limit breaks the topology of Σ, it is interesting to study the limit of the classical Gauss-Bonnet formula with respect to h^ε as $\varepsilon \to 0$, which is the subject of this paper. Our inspiration to carry out this approach are the papers [2, 3], where partial sub-Riemannian Gauss-Bonnet results for surfaces embedded in the Heisenberg group are obtained. We also refer to [7, 9] for previous results relating topology to sub-Riemannian invariants.

Let us explain the ingredients in our main result. Assume that both M and E are orientable, and choose an orientation of E. Then there exists a unique positively oriented contact form α satisfying $E = \ker \alpha$ and $\|d\alpha\| = 1$. Let $\Sigma \subseteq M$ be a compact oriented C^2-subsurface with the characteristic set

$$\operatorname{char}(\Sigma) = \{x \in \Sigma \; : \; T_x\Sigma = E_x\}.$$

This set will be contained in a one-dimensional C^1-submanifold of Σ as shown in [4, Lemma 2.4], and does not depend on the metric g_E. Let h denote the restriction to Σ of $g = g^1$ with area form $\sigma = \sigma^1$, and let β be the restriction of the contact form α. We can then define a function $a : \Sigma \to \mathbb{R}$ by

$$d\beta = -a\sigma.$$

In other words, the function a encodes the relationship between the exterior derivative on Σ with its area form. This relationship can also be expressed as $a = \cos \vartheta$, where ϑ is the oriented angle between the planes in the subbundle E and the tangent bundle $T\Sigma$ of the surface. The points $x \in \Sigma$ with $|a(x)| = 1$ correspond to the set $\operatorname{char}(\Sigma)$ where by definition the oriented angle has to be either 0 or π. In particular, since a is a smooth function taking values in the interval $[-1, 1]$, any point in the characteristic set is a critical point of a. For a final interpretation of the function a, we also mention that, up to a sign, a equals the divergence of the characteristic vector field of the foliation of $T\Sigma \cap E$ in the sense of [12, Chapter 4.6], see Remark 2.4 for details.

In studying the geometry of (Σ, h^ε) as ε changes, we will observe that the change in geometric invariants are controlled by the function a and its derivatives. As a consequence, we will need some assumptions when it comes to the decay of a close to the characteristic set.

$$\begin{gathered}\text{There exists } \delta > 0, \text{ such that the derivatives} \\ \frac{d}{dc}\int_{0\leq 1-a\leq c}\sigma \text{ and } \frac{d}{dc}\int_{0\leq 1+a\leq c}\sigma \\ \text{exist and are uniformly bounded on } c\in(0,\delta).\end{gathered} \tag{A}$$

One can get an intuition for this condition by observing that for $\xi \in \mathbb{R}^2$,

$$\frac{d}{dc}\int_{0\leq\|\xi\|^s\leq c} d\xi = \frac{2\pi}{s}c^{(2-s)/s}, \tag{1.1}$$

which is uniformly bounded for c small when $0 < s \leq 2$ but not for $s > 2$. Observe furthermore that if $x_0 \in \operatorname{char}(\Sigma)$ is an isolated characteristic point with $a(x_0) = \pm 1$, then it will be a critical point of a, and so $a(x) = \pm 1 + O(d_h(x_0, x)^2)$ close to x_0. It is therefore no surprise that if all the points in $\operatorname{char}(\Sigma)$ are isolated and non-degenerate critical points of a, then condition (A) holds as we get the decay in volume similar to the case $s = 2$ in (1.1). See Section 2.5 for a detailed proof. The condition of non-degeneracy can be seen as an analogue of previous results [19] and [12, Chapter 4.2] where the non-degeneracy condition is in terms of respectively a level set function and the characteristic vector field. See also Section 2.4, where we show that (A) does hold if the level sets of a are 1-dimensional submanifolds whose length do not decrease slower than $\sqrt{1-|a|}$ when approaching the characteristic set.

Outside the characteristic set, i.e., on $0 < |a| < 1$, the intersection $T\Sigma \cap E$ is a one-dimensional subbundle and it is possible to uniquely define an h-unit vector field X on Σ spanning this intersection. We then also get a corresponding positively oriented orthonormal basis X, X_2, which, when added taken along with the Reeb vector field Z, gives us a basis $X(x), X_2(x), Z(x)$ for T_xM for every $x \in \Sigma \setminus \operatorname{char}(\Sigma)$. We can then define the following curvature invariant on $\Sigma \setminus \operatorname{char}(\Sigma)$ by

$$\begin{aligned} K_{\Sigma,E} = &-\left(\operatorname{div}(X_2) - \sqrt{1-a^2}\langle[Z,X],X\rangle\right)\frac{X_2 a}{a} \\ &-\left(1+\frac{\sqrt{1-a^2}}{a}\operatorname{div}(X)\right)^2 - 1 + a^2 - \frac{\sqrt{1-a^2}}{a}\operatorname{div}(X). \end{aligned} \tag{1.2}$$

The divergence above is with respect to the volume form σ. The function $K_{\Sigma,E}$ does not depend on the orientation of E, and we will show that it is uniformly bounded on $\Sigma \setminus \operatorname{char}(\Sigma)$. For the geometric interpretation of $K_{\Sigma,E}$, note that the space (Σ, h^ε) converges towards a one-dimensional object [4, 6]. The one-dimensional foliation tangent to $T\Sigma \cap E$ will have a decreasing geodesic curvature with $k_E^0 := \lim_{\varepsilon\to 0}\frac{1}{\sqrt{\varepsilon}}k_E^\varepsilon$ describing its rate of convergence to zero at the limit. We can then express $K_{\Sigma,E}$ in terms of this limit and the angle $a = \cos\vartheta$ by

$$K_{\Sigma,E} = \cos^2\vartheta + \frac{\sin^2\vartheta}{\cos\vartheta}(k_E^0 - \langle [Z, X], X\rangle)X_2\vartheta + (1 - X\vartheta)X\vartheta.$$

Recall that X denotes a derivative along the foliation $T\Sigma \cap E$, which is the only permitted direction of travel in the limit, while X_2 is the oriented normal direction of this foliation. If $\mathcal{L}_Z$ denotes the Lie derivative, then the term $\langle [Z, X], X\rangle = -\frac{1}{2}(\mathcal{L}_Z g^\varepsilon)(X, X)$ vanishes when the Reeb vector field is an infinitesimal isometry of (E, g), and so $\langle [Z, X], X\rangle$ measures how far we are from this property to hold along $T\Sigma \cap E$. We will use the invariant $K_{\Sigma,E}$ for our main result.

Theorem 1.1 (Sub-Riemannian Gauss-Bonnet theorem) *Let $\Sigma \subseteq M$ be a compact C^2-surface without boundary such that Assumption (A) holds. Then*

$$\left.\frac{d}{dc}\right|_{c=0} \int_{|a|>1-c} K_{\Sigma,E}(x)\sigma(x) = 2\pi\chi(\Sigma).$$

In particular, the Euler characteristic $\chi(\Sigma)$ can be determined by the values of $K_{\Sigma,E}$ in a neighborhood of $\mathrm{char}(\Sigma)$. We get the following simple corollary

Corollary 1.2 (a) *If* $\mathrm{char}(\Sigma) = \emptyset$, *then* $\chi(\Sigma) = 0$.
(b) *If $K_{\Sigma,E}$ is non-negative in a neighborhood of* $\mathrm{char}(\Sigma)$, *then Σ is homeomorphic to a sphere or a torus.*

If $\mathrm{char}(\Sigma)$ consists of isolated, non-degenerate critical points of a and with limits $K_{\Sigma,E}(x) = \lim_{y\to x} K_\Sigma(y)$ existing at all of these points, then

$$\chi(\Sigma) = \sum_{x\in\mathrm{char}(\Sigma)} \frac{K_{\Sigma,E}(x)}{\sqrt{\det(\nabla^{h,2}a(x))}}, \tag{1.3}$$

where $\nabla^{h,2}a$ is the Hessian with respect to h; see Section 2.5 for details.

1.1 Structure of the paper.

In Section 2, we give the basic definitions related to sub-Riemannian 3-dimensional contact manifolds (M, E, g) with a surface $\Sigma \subseteq M$. We will also give sufficient conditions for assumption (A) to hold in Section 2.4 and Section 2.5. We provide a worked example in Section 2.6 which illustrates how the Euler characteristic can be computed with (1.3). In Section 3 we show how curvature of the taming Riemannian metric g^ε for the ambient space M varies with ε. We continue by looking at the geometry of the metric $h^\varepsilon = g^\varepsilon|_{T\Sigma}$ in Section 4. In Section 5 we give the proof of Theorem 1.1 in several steps. With some minor restrictions on the boundary, we also give a Gauss-Bonnet theorem for surfaces with a piecewise C^2-boundary in Section 6.

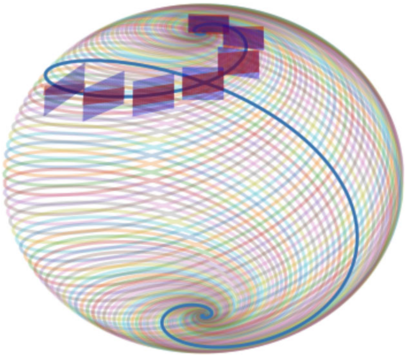

Fig. 1 The figure shows the foliation of curves tangent to $\Sigma = S^2$ and the contact distribution $E = \ker\{dz + (y/2dx - x/2dy)\}$. Along the highlighted blue curve, we have shown the tangent plane in blue, while the planes of E are in red. The angle $\vartheta = \cos^{-1} a$ is the angle between these planes. These coincide at the north and south pole, which make up $\mathrm{char}(\Sigma)$. We are studying the Gauss-Bonnet theorem under a metric h^ε which is constant for the vector field X tangent to the shown foliation, while the length of its 90 degree rotation X_2 approaches infinity as $\varepsilon \to 0$. For details on this example, see Section 2.6 (Color figure online)

1.2 Relation to previous works

Let K^ε denote the Gaussian curvature of Σ with respect to h^ε, and let σ^ε be its volume form. If we consider the limit of the equation $\int_\Sigma K^\varepsilon d\sigma^\varepsilon = 2\pi\chi(\Sigma)$ as $\varepsilon \to 0$, then away from the characteristic set, $K^\varepsilon d\sigma^\varepsilon$ only has terms of half-integer order with respect to ε, starting with order $-1/2$, see Section 5 for details. Isolating this part of order $-1/2$, and using that $\sqrt{\varepsilon}\int_\Sigma K^\varepsilon d\sigma^\varepsilon = 0$, we obtain an identity for a part of the curvature that has vanishing average over Σ. This observation is found in our paper in Remark 5.3. Such results have appeared previously for given ambient contact manifolds such as Heisenberg group and $\mathrm{SE}(2)$, see [2, 11, 14, 16, 17, 21] for examples. These examples include assumptions to avoid any contribution from the characteristic set $\mathrm{char}(\Sigma)$. For example, in [2], the surface Σ is defined locally as the level set of a function u, and it is assumed that $\frac{1}{\|\nabla^E u\|}$ is integrable near the characteristic set, which in our notation is equivalent to assuming that $\frac{1}{\sqrt{1-a^2}}$ is integrable close to the set $|a| = 1$. See Section 2.3 for details.

Our paper differs from this previous work in that we are mainly interested in how the topological Euler characteristic $\chi(\Sigma)$ is preserved under the limit. We are hence interested in the term of order zero of the integral $\int_\Sigma K^\varepsilon d\sigma^\varepsilon$, which can only be obtained through the study of the integral close to $\mathrm{char}(\Sigma)$. We also have no restriction on the ambient sub-Riemannian contact manifold (M, E, g), apart from assumption (A).

Finally, we also mention similar work in [1], on almost-Riemannian manifolds such as the Gruhsin plane, which can be considered as the case where h^ε approaches a Riemannian metric outside a small singular set.

2 Contact manifolds and the horizontal angle parameter

2.1 Contact distributions and the Reeb vector field

Let M be a three-dimensional manifold, with E being a rank two, contact distribution. In other words, we have $E + [E, E] = TM$. For simplicity, we will assume that both M and E are orientable and with chosen orientations. It follows that the subbundle $\operatorname{Ann} E \subseteq T^*M$ of covectors vanishing on E is orientable as well. Let E be equipped with a fiber metric g_E making (M, E, g_E) into a sub-Riemannian manifold. Let α be the unique non-vanishing section of $\operatorname{Ann}(E)$ satisfying $d\alpha(u, v) = -1$ for any positively oriented orthonormal basis $u, v \in E_x$ and any $x \in M$. *The Reeb vector field* is then the unique vector field Z satisfying

$$\alpha(Z) = 1, \qquad d\alpha(Z, \cdot) = 0. \tag{2.1}$$

We will use the Reeb vector field to extend g_E to a Riemannian metric. Consider a Riemannian metric g on M such that $g|_E = g_E$ and such that Z is orthogonal to E with $\|Z\| = 1$. We write $g = \langle \cdot, \cdot \rangle$.

Introduce a tensor $J : TM \to E$ by $u, v \in T_xM$, $x \in M$,

$$d\alpha(v, w) = \langle v, Jw \rangle = -\langle Jv, w \rangle.$$

By our definition of α it follows that J is an almost complex structure when restricted to E, corresponding to a rotation of $\frac{\pi}{2}$ in the positive direction. We finally introduce a symmetric tensor $\tau : TM \to E \subseteq TM$ by

$$\langle \tau v, w \rangle = \langle v, \tau w \rangle = \frac{1}{2}(\mathcal{L}_Z g)(\operatorname{pr}_E v, \operatorname{pr}_E w),$$

where $\mathcal{L}_Z$ denotes the Lie derivative with respcet to the Reeb vector field. We emphasize that from the previous definitions $\tau Z = JZ = 0$.

We want a connection such that both E and $E^\perp$ are parallel and with the torsion as simple as possible. We define a connection ∇ such that if Z is the Reeb vector field and Y_1 and Y_2 are arbitrary sections of E, then

$$\nabla Z = 0, \qquad \nabla_Z Y_1 = [Z, Y_1] + \tau Y_1, \qquad \nabla_{Y_1} Y_2 = \operatorname{pr}_E \nabla^g_{Y_1} Y_2,$$

where pr_E is the g-projection to E and ∇^g is the Levi-Civita connection of g. This connection is compatible with g and has torsion

$$T(V, W) = -\langle JV, W \rangle Z + \alpha(V)\tau W - \alpha(W)\tau V, \qquad V, W \in \Gamma(TM).$$

See [15, 20] for more about the choice of connection in this setting, and this connection in particular.

Remark 2.1 (a) The defining properties of the Reeb vector field Z in (2.1) imply that $[Z, E] \subseteq E$, which means that its corresponding flow e^{tZ} also preserve E. If

$\tau = 0$, then e^{tZ} will be a sub-Riemannian isometry for every t, i.e., $(e^{tZ})_*$ will map E to iself isometrically. Hence, the tensor τ can be considered as measuring how far Z deviates from being an infinitesimal isometry.

(b) We note the following identities

$$\nabla J = 0, \qquad \operatorname{tr}_g \tau = 0, \qquad \tau J = -J\tau. \tag{2.2}$$

We also observe that since $\mathcal{L}_Z d\alpha = d\mathcal{L}_Z\alpha = 0$, we have $\mathcal{L}_Z J = 2\tau J$. For the proof of these identities, see, e.g., [13].

2.2 Surfaces and the horizontal angle parameter

Let Σ be an oriented, compact surface embedded in (M, E, g_E) without boundary. Recall that α is the contact form and g is the extension to a Riemannian metric using the Reeb vector field. We write $h = g|_{T\Sigma}$ for the induced Riemannian metric on Σ, with σ being the corresponding volume form. We will use the following notation for the rest of the paper. We define *the horizontal angle parameter* $a(x)$ at $x \in \Sigma$ as the inner product between the positively oriented normal vectors of $T_x\Sigma$ and E_x. If v_1, v_2 and w_1, w_2 are positively oriented orthonormal bases of respectively $T_x\Sigma$ and E_x, then

$$a(x) = \langle v_1 \wedge v_2, w_1 \wedge w_2\rangle = -d\alpha(w_1, w_2).$$

We observe that $|a(x)| = 1$ exactly at $\operatorname{char}(\Sigma)$, the set of points $x \in \Sigma$ where $T_x\Sigma = E_x$. Write

$$\Sigma_{c_1 \le c_2} = \{x \in \Sigma : c_1 \le a(x) \le c_2\}, \quad \Sigma_c = \Sigma_{c\le c}, \quad \Sigma' = \Sigma \setminus \operatorname{char}(\Sigma). \tag{2.3}$$

Lemma 2.2 *Let $\beta = \alpha|_{T\Sigma}$ be the restriction of the contact form to Σ.*

(a) *We have identity $d\beta = -a\sigma$ and $\|\beta\| = \sqrt{1-a^2}$.*

(b) *There exists a unique unit vector field X on Σ' with values in $T\Sigma' \cap E|_{\Sigma'}$ such that*

$$\sqrt{1-a^2}\,\iota_X\sigma = \beta.$$

Furthermore, the unique vector field X_2 such that X, X_2 is a positive orthonormal basis is given by

$$X_2 = \sqrt{1-a^2}Z + aJX.$$

Proof (a) This result follows directly from the definition of a.

(b) Since β is non-vanishing on Σ', we obtain that $\ker\beta$ is a one-dimensional line-bundle spanned by some unit vector field X, uniquely determined up to sign. As $\iota_X\sigma$ is a non-vanishing one-form that vanishes on $\operatorname{span}\{X\}$, it follows that $\iota_X\sigma = \varphi_1\beta$ for some non-vanishing function φ_1. By requiring $\varphi_1 > 0$, the choice of X is unique. Furthermore, if X, X_2 is a positively oriented orthonormal basis, then

$$\sigma(X, X_2) = 1 = (\iota_X\sigma)(X_2) = \varphi_1\beta(X_2), \qquad d\beta(X, X_2) = -a.$$

It follows that $X_2 = \frac{1}{\varphi_1} Z + aJX$ and since X_2 must have length 1, we finally have $\frac{1}{\varphi_1} = \sqrt{1-a^2}$.

□

To simplify notations later, we will also introduce the following functions. Define $\tau_0, \tau_1 : \Sigma' \to \mathbb{R}$ by $\tau_0 = \langle \tau X, X\rangle = -\langle \tau JX, JX\rangle$ and $\tau_1 = \langle \tau X, JX\rangle$. We remark the following relationship.

Lemma 2.3

$$\frac{Xa}{\sqrt{1-a^2}} = a^2 + \sqrt{1-a^2}\langle \nabla_{X_2} X, JX\rangle - (1-a^2)\tau_1. \tag{2.4}$$

Proof We consider the normal vector $N = aZ - \sqrt{1-a^2}JX$. Since $T\Sigma$ is Frobenius integrable, we have $\langle N, [X, X_2]\rangle = 0$. The bracket is

$$[X, X_2] = -\tfrac{aXa}{\sqrt{1-a^2}} Z + XaJX + \sqrt{1-a^2}[X, Z] + a[X, JX]$$

and so

$$\begin{aligned} 0 = \langle N, [X, X_2]\rangle &= -\tfrac{a^2 Xa}{\sqrt{1-a^2}} + a^2 - \sqrt{1-a^2}Xa \\ &\quad + \sqrt{1-a^2}\langle \sqrt{1-a^2}[Z, X] + a[JX, X], JX\rangle \\ &= -\tfrac{Xa}{\sqrt{1-a^2}} + a^2 + \sqrt{1-a^2}\langle \nabla_{X_2} X, JX\rangle - (1-a^2)\langle \tau X, JX\rangle. \end{aligned}$$

The result follows. □

Remark 2.4 *(Characteristic vector fields)* Let us compare this basis X, X_2 to previous work in [12, Chapter 4.6], see also [5]. *A characteristic vector field* on Σ is a vector field $\tilde{X}$ with values in $E \cap T\Sigma$ that vanishes on $\mathrm{char}(\Sigma)$, while also satisfying

$$\mathrm{div}(\tilde{X})(x) \neq 0, \qquad x \in \mathrm{char}(\Sigma).$$

We get such a globally defined vector field by $\tilde{X} = \sqrt{1-a^2}X$, which will then satisfy $\iota_{\tilde{X}}\sigma = \beta$ and

$$\mathrm{div}(\tilde{X})\sigma = d\iota_{\tilde{X}}\sigma = d\beta = -a\sigma.$$

Hence the horizontal angle parameter can be seen as the negative divergence of this characteristic vector field $\tilde{X}$. Writing X as $X = \frac{1}{\sqrt{1-a^2}}\tilde{X}$, we have

$$\mathrm{div}(X) = \frac{a}{\sqrt{1-a^2}}\left(\frac{Xa}{\sqrt{1-a^2}} - 1\right).$$

If we write $a = \cos\vartheta$, then $\mathrm{div}(X) = -\cot\vartheta(1 + X\vartheta)$

2.3 Local description

Working locally, we may assume that $\Sigma = u^{-1}(0)$ is the level set of a function $u : M \to \mathbb{R}$ and that E has a local, positively oriented, orthonormal basis Y_1 and Y_2. Let ∇u and $\nabla^E u$ be respectively the Riemannian and sub-Riemannian gradient, i.e., $\nabla^E u = Y_1 u Y_1 + Y_2 u Y_2$ and $\nabla u = \nabla^E u + Zu Z$. We then observe that $T\Sigma$ is spanned by orthonormal basis

$$\begin{aligned} X &= \frac{1}{\|\nabla^E u\|}(-(Y_2 u)Y_1 + (Y_1 u)Y_2), \\ JX &= \frac{1}{\|\nabla^E u\|}(-(Y_1 u)Y_1 - (Y_2 u)Y_2), \\ X_2 &= \frac{1}{\|\nabla u\|\|\nabla^E u\|}\left(\|\nabla^E u\|^2 Z - (Zu)(Y_1 u)Y_1 - (Zu)(Y_2 u)Y_2\right), \end{aligned}$$

with X being in E. We give Σ an orientation by defining X, X_2 to be positively oriented. We then see that

$$-d\alpha(X, X_2) = a = \frac{Zu}{\|\nabla u\|}. \tag{2.5}$$

and hence $\sqrt{1-a^2} = \frac{\|\nabla^E u\|}{\|\nabla u\|}$. Replacing u with $U = \frac{1}{|\nabla u|}u$, we have $U^{-1}(0) = \Sigma$, but now $a = ZU$ and $\sqrt{1-a^2} = \|\nabla^E U\|$. It follows that

$$\begin{aligned} K_{\Sigma,E} = &-\left(\langle [X, X_2], X\rangle - \|\nabla^E U\|\langle [Z, X], X\rangle\right)\frac{X_2 ZU}{ZU} \\ &-\left(1 + \frac{\|\nabla^E U\|}{ZU}\langle [X_2, X], X_2\rangle\right)^2 - \|\nabla^E U\|^2 - \frac{\|\nabla^E U\|}{ZU}\langle [X_2, X], X_2\rangle, \end{aligned}$$

2.4 A sufficient condition for (A)

We present the following sufficient condition for our Assumption (A), in terms of the geometry of the level sets of a. Recall the definition of Σ_c and $\Sigma_{c_1 \le c_2}$ in (2.3).

Proposition 2.5 *Assume that the following two assumptions hold.*

(i) *1 is an isolated critical value of $|a|$, i.e., for some $0 < c_1 < 1$, $\nabla^h a|_x \neq 0$ whenever $c_1 < |a(x)| < 1$.*
(ii) *For Σ_c as in (2.3) with length $\ell(\Sigma_c)$, we have asymptotics*

$$\ell(\Sigma_c) = O(\sqrt{1-|c|}), \quad as \quad c \to \pm 1.$$

Then (A) holds.

Proof For proving (A), it is sufficient to prove that for some $\delta > 0$, $\frac{d}{dc}\int_{\Sigma_{-1\leq -1+c}} a\sigma$ and $\frac{d}{dc}\int_{\Sigma_{1-c\leq 1}} a\sigma$ exists and are uniformly bounded for $(0, \delta)$. Define $\Phi(c) = \int_{\Sigma_{0\leq c}} a\sigma$ and $\Phi(-c) = -\int_{\Sigma_{-c\leq 0}} a\sigma$ for $c \geq 0$. Recall that $d\theta = -a\sigma$. By (i),

$$a : \Sigma \to [-1, 1]$$

is a submersion on $a^{-1}(c_1, 1)$ which is proper by the compactness of Σ. It is a fiber bundle by the Ehresmann theorem, and hence we can use a as a coordinate on $a^{-1}(c_1, 1)$, and consider the fibers to be diffeomorphic to a one-dimensional manifold F. Furthermore, write

$$a\sigma = \nu \wedge da, \qquad \text{with} \qquad \nu = \frac{a}{\|\nabla^h a\|^2}((X_2 a)X^* - (Xa)X_2^*).$$

We then have that $\Phi(1) - \Phi(c_1) = \int_{c_1}^1 \int_{y\in F} \nu(a, y)da$, and hence $\Phi'(c) = \int_{y\in F} \nu(c, y)$. Next, we need to show that $\Phi'(c)$ is uniformly bounded on $(c_1, 1)$. Since $\Phi'(c) > 0$, it is sufficient to show that $\limsup_{c\to 1} \frac{\Phi(1)-\Phi(c)}{1-c} < \infty$. We observe that since $|\beta| = \sqrt{1-a^2}$,

$$\Phi(1) - \Phi(c) = \int_{\Sigma_{c\leq 1}} a\sigma = -\int_{\Sigma_{c\leq 1}} d\beta = -\int_{\Sigma_c} \beta \leq \sqrt{1-c^2}\,\ell(\Sigma_c),$$

it follows by (ii) that

$$\limsup_{c\to 1} \frac{\tilde{\Phi}(1) - \tilde{\Phi}(c)}{1-c} \leq \limsup_{c\to 1} \ell(\Sigma_c)\frac{\sqrt{1+c}}{\sqrt{1-c}} < \infty.$$

The proof is completed by repeating this argument for the intervall $(-1, -c_1)$. □

2.5 Result for isolated points

We consider the special case where the characteristic set only consist of isolated, non-degenerate critical points of a. Recall that $\nabla^{h,2}a$ denotes the Hessian with respect to the Levi-Civita connection ∇^h, i.e.

$$\nabla^{h,2}_{X,Y} = (\nabla^h_X da)(Y) = (XY - \nabla^h_X Y)a.$$

Theorem 2.6 *Assume that* $\operatorname{char}(\Sigma) = a^{-1}(\pm 1)$ *consists of isolated points, and that*

$$\det(\nabla^{h,2}a(x)) \neq 0, \qquad \textit{for any } x \in \operatorname{char}(\Sigma).$$

Then condition (A) *holds. Furthermore, if* $K_{\Sigma,E}$ *has well defined limits* $K_{\Sigma,E}(x) := \lim_{y\to x} K_{\Sigma,E}(y)$ *for every* $x \in \mathrm{char}(\Sigma)$, *then*

$$\chi(\Sigma) = \sum_{x\in\mathrm{char}(\Sigma)} \frac{K_{\Sigma,E}(x)}{\sqrt{\det(\nabla^{h,2}a(x))}}.$$

Proof Since $\mathrm{char}(\Sigma) = a^{-1}(\pm 1)$ is a closed subset of a compact manifold, it can only be without any accumulation point if it is a finite set. Hence, assuming that points are isolated is equivalent to $\mathrm{char}(\Sigma)$ being finite. Let $x \in \mathrm{char}(\Sigma)$ be an arbitrary point in the characteristic set and let U_x be a neighborhood of x such that $\{x\} = U_x \cap \mathrm{char}(\Sigma)$. Since x has to be a maximum or minimum of a and since it is non-degenerate, we must have $\det(\nabla^{h,2}a) > 0$. By possibly shrinking U_x, we may assume that $\nabla^h a \neq 0$ on $U_x \setminus \{x\}$, meaning in particular that the level sets of a intersected with U_x are 1-dimensional submanifold outside of $a = 1$.

Without loss of generality, we assume that $a(x) = 1$, as the case $a(1) = -1$ can be completed in a similar manner. Define a function

$$\Phi_x(c) = \int_{U_x\cap\{a(x)\le c\}} \sigma.$$

As each preimage $a^{-1}(c) \cap U_x$ is a one-dimensional submanifold, we can use the argument of Proposition 2.5 to show that $\Phi_x(c)$ is differentiable for $c < 1$. Working locally around x, define

$$\Delta_{x,s}\Phi = \Phi_x(1) - \Phi_x(1-s).$$

Close to x, define a local coordinate system (ξ_1, ξ_2) created using the Riemannian exponenital map and the unit vectors in the in the directions of the eigenvectors of $\nabla^{h,2}a(x)$, and with corresponding eigenvalues $-\lambda_1 < 0$ and $-\lambda_2 < 0$. As the coordinate derivatives are orthonormal at x, we have $\sigma = (1 + O(|\xi|))d\xi_1 \wedge d\xi_2$. Then close to x,

$$a(\xi_1, \xi_2) = 1 - \lambda_1\xi_1^2/2 - \lambda_2\xi_2^2/2 - O(|\xi|^3)$$

For this reason, for some constant $C > 0$ and for small enough δ, on the set $|\xi| < \delta$,

$$(\lambda_1/2 - C\delta)\xi_1^2 + (\lambda_2/2 - C\delta)\xi_2^2 \le 1 - a \le (\lambda_1/2 + C\delta)\xi_1^2 + (\lambda_2/2 + C\delta)\xi_2^2$$

and for $\delta < \min\{\lambda_1, \lambda_2\}/(2C)$

$$\int_{(\lambda_1+C\delta)\xi_1^2+(\lambda_2+C\delta)\xi_2^2\le s} \sigma \le \Delta_{x,s}\Phi \le \int_{(\lambda_1-C\delta)\xi_1^2+(\lambda_2-C\delta)\xi_2^2\le s} \sigma.$$

Since $(\lambda_1/2 \pm C\delta)\xi_1^2 + (\lambda_2/2 \pm C\delta)\xi_2^2 \le s$ is the interior and boundary of the ellipse with axes $\sqrt{\frac{s}{\lambda_1/2 \pm C\delta}}$ and $\sqrt{\frac{s}{\lambda_2/2 \pm C\delta}}$, we obtain

$$\frac{2\pi s}{\sqrt{(\lambda_1 + 2C\delta)(\lambda_2 + 2C\delta)}} + O(s^{3/2}) \le \Delta_{x,s}\Phi \le \frac{2\pi s}{\sqrt{(\lambda_1 - 2C\delta)(\lambda_2 - 2C\delta)}} + O(s^{3/2}).$$

Dividing with s and letting both s and δ approach zero, we obtain that

$$\Phi_x'(1) = \frac{2\pi}{\sqrt{\lambda_1 \lambda_2}}.$$

That this identity is satisfied for all the points in $\mathrm{char}(\Sigma)$ implies (A). Furthermore, we see that if $K_{\Sigma,E}$ has a limit at x, then

$$\frac{d}{dc}\bigg|_{c=0} \int_{U_x \cap \{1-a<c\}} K_{\Sigma,E}\sigma = \frac{2\pi}{\sqrt{\lambda_1\lambda_2}} K(x).$$

The result follows. □

Remark 2.7 We mention that [12, Prop 4.6.11] guarantees the existence of a C^∞-close perturbation $\tilde{\Sigma}$ of our surface Σ such that the characteristic foliation is Morse-Smale type, the latter of which only have isolated points in the characteristic set.

Remark 2.8 In Section 2.6, we show an example where the limit of $K_{\Sigma,E}$ exists on the characteristic set, but it is unclear if we can always assume that such limits holds. Our main result in Theorem 1.1 considers limits of the Gaussian curvature K^ε with respect to a varying Riemannian metric h^ε as $\varepsilon \to 0$. While K^ε is a continuous on all points, the definition of $K_{\Sigma,E}$ involves limits of terms involving $\sqrt{\frac{\varepsilon}{1-(1-\varepsilon)a^2}}$, which approaches 0 outside $\mathrm{char}(\Sigma)$, but 1 on the characteristic set, see Section 5 for details.

2.6 Example: The Euclidean unit sphere in the Heisenberg group

We consider the Heisenberg group as $M = \mathbb{R}^3$ where E has a positively oriented orthonormal basis $A = \partial_x - \frac{1}{2}y\partial_z$ and $B = \partial_y + \frac{1}{2}x\partial_z$. The corresponding Reeb vector field is $Z = \partial_z$. For this case, we can verify that $\tau = 0$ globally. We orient E so that the basis A, B is positively oriented, and hence $JA = B$ and $JB = -A$. The contact form can then be written as $\alpha = dz + \frac{1}{2}(ydx - xdy)$.

Inside the Heisenberg group, we consider Σ as the Euclidean sphere which is a level set of $u = x^2 + y^2 + z^2 - 1$. It follows from (2.5) that

$$\nabla^E u = (2x - yz)A + (2y + xz)B =: q_1 A + q_2 B, \quad \nabla u = \nabla^E u + 2zZ,$$
$$a = \frac{2z}{\sqrt{4z^2 + q_1^2 + q_2^2}} = \frac{2z}{\sqrt{4 + (x^2 + y^2)z^2}}.$$

Outside the equator we can use x, y as coordinates and introduce $q_0 = x^2 + y^2$. In these coordinates, $z = s\sqrt{1-q_0}$ where s has respective values 1 or -1 depending on which hemisphere we are on. Furthermore,

$$q_1^2 + q_2^2 = q_0(5-q_0), \qquad a = 2s\sqrt{\frac{1-q_0}{4+q_0-q_0^2}}, \qquad \sqrt{1-a^2} = \sqrt{\frac{q_0(5-q_0)}{4+q_0-q_0^2}}.$$

The restriction of the contact form is written

$$\beta = -\frac{s}{\sqrt{1-q_0}}(xdx + ydy) + \frac{1}{2}(ydx - xdy) = -\frac{1}{2z}(q_1 dx + q_2 dy),$$

and the area form is $\sigma = -\frac{1}{a}d\beta = \frac{1}{a}dx \wedge dy = \frac{1}{a}dx \wedge dy = \frac{s}{2}\sqrt{\frac{4+q_0-q_0^2}{1-q_0}}dx \wedge dy$. The local coordinate derivatives ∂_x and ∂_y on the surface are then the restriction of $A - \frac{q_1}{2z}Z$ and $B - \frac{q_2}{2z}Z$ from the ambient space, which allows us to express the metric as

$$\begin{pmatrix} h_{xx} & h_{xy} \\ h_{xy} & h_{yy} \end{pmatrix} = \begin{pmatrix} 1 + \frac{q_1^2}{4z^2} & \frac{q_1 q_2}{4z^2} \\ \frac{q_1 q_2}{4z^2} & 1 + \frac{q_2^2}{4z^2} \end{pmatrix}, \qquad \det h = 1 + \frac{q_1^2 + q_2^2}{4z^2} = \frac{1}{a^2}.$$

From the identity $\sqrt{1-a^2}\iota_X\sigma = \frac{\sqrt{1-a^2}}{a}\iota_X dx \wedge dy = \beta$ and the requirement that X, X_2 should be a positive orthonormal basis,

$$\begin{aligned} X &= \frac{1}{\sqrt{q_0(5-q_0)}}(-q_2\partial_x + q_1\partial_y), \\ X_2 &= \frac{-2z}{\sqrt{q_0(5-q_0)}\sqrt{4z^2 + q_1^2 + q_2^2}}(q_1\partial_x + q_2\partial_y), \end{aligned}$$

or with the notation $Y_0 = x\partial_x + y\partial_y$ and $Y_1 = y\partial_x - x\partial_y$,

$$\begin{aligned} X &= \frac{1}{\sqrt{q_0(5-q_0)}}(\sqrt{1-q_0}Y_0 - 2Y_1), \\ X_2 &= \frac{-a}{\sqrt{q_0(5-q_0)}}(2Y_0 - \sqrt{1-q_0}Y_1). \end{aligned}$$

Using that $(x, y) = (0, 0)$ is a critical point of a and that ∂_x, ∂_y is an orthonormal basis when $z = 0$, we obtain that

$$\det(\nabla^{h,2}a)(0,0) = \det\begin{pmatrix} \partial_x^2 a & \partial_x\partial_y a \\ \partial_y\partial_x a & \partial_y^2 a \end{pmatrix}\Bigg|_{x=0,y=0} = \det\begin{pmatrix} -5s/4 & 0 \\ 0 & -5s/4 \end{pmatrix} = (5/4)^2.$$

Hence, condition (A) is satisfied by Theorem 2.6. For the computation of the divergence, we observe that since $\sqrt{\det h} = \frac{1}{|a|}$ and since the poles are critical points of a, then for any vector field $\tilde{Y} = \tilde{Y}^1\partial_x + \tilde{Y}^2\partial_y$,

$$\lim_{(x,y)\to 0} \operatorname{div}(\tilde{Y}) = \lim_{(x,y)\to 0} |a| \left(\partial_x \frac{\tilde{Y}^1}{|a|} + \partial_y \frac{\tilde{Y}_2}{|a|} \right) = \lim_{(x,y)\to 0} (\partial_x \tilde{Y}^1 + \partial_y \tilde{Y}_2).$$

We compute

$$\lim_{(x,y)\to 0} \sqrt{q_0}\operatorname{div}(X) = \frac{1}{\sqrt{5}} \lim_{(x,y)\to 0} \sqrt{q_0} \left(\frac{-\partial_x q_2 + \partial_y q_1}{\sqrt{q_0}} - \frac{-xq_2 + yq_1}{(\sqrt{q_0})^3} \right) = -\frac{s}{\sqrt{5}}.$$

and with similar calculation

$$\lim_{(x,y)\to 0} \sqrt{q_0}\operatorname{div}(X_2) = -\frac{s}{\sqrt{5}} \lim_{(x,y)\to 0} \sqrt{q_0} \left(\frac{\partial_x q_1 + \partial_y q_2}{\sqrt{q_0}} - \frac{xq_1 + yq_2}{(\sqrt{q_0})^3} \right) = -\frac{2s}{\sqrt{5}}.$$

Furthermore, $\lim_{(x,y)\to 0} \sqrt{\frac{1-a^2}{q_0}} = \frac{\sqrt{5}}{2}$ and

$$\lim_{(x,y)\to 0} \frac{X_2 a}{a\sqrt{q_0}} = -\frac{2}{\sqrt{5}} \lim_{(x,y)\to 0} \frac{Y_0 a}{q_0} = -\frac{2}{\sqrt{5}} \cdot \frac{s\cdot(-5)\cdot 2}{2\cdot 4^2} = \frac{\sqrt{5}}{2}s.$$

In the latter equation, we have used that $Y_1 q_0 = 0$ and $Y_0 q_0 = 2q_0$. Inserting these identities into (1.2), then the limits $K_{\Sigma,E}$ at the poles are

$$\begin{aligned} &\lim_{(x,y)\to 0} \left(-\operatorname{div}(X_2)\cdot \frac{X_2 a}{a} - \left(1 + \frac{\sqrt{1-a^2}}{a} \operatorname{div}(X) \right)^2 - \frac{\sqrt{1-a^2}}{a} \operatorname{div}(X) \right) \\ &= 1 - \frac{1}{4} + \frac{1}{2} = \frac{5}{4}. \end{aligned}$$

Hence, the ratio $K_{\Sigma,E}/\sqrt{\det \nabla^{h,2} a}$ equals 1 at both points in $\operatorname{char}(\Sigma)$ and we can then recover the Euler characteristics as $\chi(\Sigma) = 1 + 1 = 2$.

3 Geometric identities and the variational metric

3.1 Variational metrics and the Levi-Civita connection

We will now define a family of Riemannian metrics g^ε on M such that $g^\varepsilon|_E = g_E$ and such that Z is orthogonal to E with

$$g^\varepsilon(Z, Z) = \langle Z, Z\rangle_\varepsilon = \tfrac{1}{\varepsilon}.$$

Remark that $\varepsilon = 1$ corresponds to our case in Section 2. The tensors J and τ are still skew-symmetric and symmetric respectively relative to these metrics. Furthermore, if

∇ is as in Section 2.1, then this connection is compatible with g^ε for every $\varepsilon > 0$. Our goal first will be to relate the geometry of (M, g^ε) using the tensors J and τ. For each g^ε, we have a corresponding Levi-Civita connection ∇^ε.

Lemma 3.1 *Introduce an operator*

$$Q_\varepsilon = \tfrac{1}{2}J - \varepsilon\tau.$$

Then for arbitrary vector fields W_1 and W_2 in $\Gamma(TM)$,

$$\nabla^\varepsilon_{W_1} W_2 = \nabla_{W_1} W_2 + \langle Q_\varepsilon W_1, W_2\rangle Z - \frac{1}{\varepsilon}\alpha(W_2) Q_\varepsilon W_1 - \frac{1}{2\varepsilon}\alpha(W_1) J W_2. \quad (3.1)$$

Proof This follows from simple application of the Koszul formula. If $Y_0, Y_1, Y_2 \in \Gamma(E)$, then

$$\begin{aligned}
\langle \nabla^\varepsilon_{Y_0} Y_1, Y_2\rangle_\varepsilon &= \langle \nabla_{Y_0} Y_1, Y_2\rangle_\varepsilon, & \langle \nabla^\varepsilon_{Y_0} Z, Z\rangle_\varepsilon &= 0,\\
\langle \nabla^\varepsilon_{Y_0} Y_1, Z\rangle_\varepsilon &= \tfrac{1}{2\varepsilon}\langle [Y_0, Y_1], Z\rangle - \langle \tau Y_0, Y_1\rangle, & \langle \nabla^\varepsilon_Z Y_1, Z\rangle_\varepsilon &= -\langle \nabla^\varepsilon_Z Z, Y_1\rangle_\varepsilon = 0\\
\langle \nabla^\varepsilon_{Y_0} Z, Y_1\rangle_\varepsilon &= \langle \tau Y_0, Y_1\rangle - \tfrac{1}{2\varepsilon}\langle [Y_0, Y_1], Z\rangle, & \langle \nabla^\varepsilon_Z Z, Z\rangle_\varepsilon &= 0.
\end{aligned}$$

and $\langle \nabla^\varepsilon_Z Y_1, Y_2\rangle_\varepsilon = \langle [Z, Y_1], Y_2\rangle + \langle \tau Y_1, Y_2\rangle - \frac{1}{2\varepsilon}\langle [Y_1, Y_2], Z\rangle$. The result follows. □

3.2 Expansion of the curvature

We want to see how the curvature tensor of ∇^ε changes with respect to ε, described by the relations below.

Lemma 3.2 *If Y be a vector field with values in E. We then have identities*

$$\begin{aligned}
\langle R^\varepsilon(Y, JY)JY, Y\rangle_\varepsilon &= \langle R(Y, JY)JY, Y\rangle - \frac{3}{4\varepsilon}\|Y\|^4 + \frac{\varepsilon}{2}\|\tau\|^2\|Y||^4,\\
\langle R^\varepsilon(Y, JY)Y, Z\rangle_\varepsilon &= \langle (\nabla_{JY}\tau)Y, Y\rangle - \langle (\nabla_Y \tau)JY, Y\rangle,\\
\langle R^\varepsilon(Y, Z)Z, Y\rangle_\varepsilon &= \frac{1}{4\varepsilon^2}\|Y\|^2 - \|\tau Y\|^2 - \langle (\nabla_Z \tau)Y, Y\rangle - \frac{1}{\varepsilon}\langle \tau Y, JY\rangle,\\
\langle R^\varepsilon(Y, Z)JY, Z\rangle_\varepsilon &= -\frac{1}{\varepsilon}\langle \tau Y, Y\rangle + \langle (\nabla_Z \tau)Y + \tau^2 Y, JY\rangle.
\end{aligned}$$

Proof For general vector fields V_j, $j = 1, 2, 3, 4$, write $\eta^\varepsilon_{V_1} V_2 := \nabla^\varepsilon_{V_1} V_2 - \nabla_{V_1} V_2$. Then

$$R^\varepsilon(V_1, V_2) - R(V_1, V_2) = (\nabla_{V_1}\eta^\varepsilon)_{V_2} - (\nabla_{V_2}\eta^\varepsilon)_{V_1} + \eta^\varepsilon_{T(V_1, V_2)} + [\eta^\varepsilon_{V_1}, \eta^\varepsilon_{V_2}].$$

From Lemma 3.1, we have $\eta^\varepsilon_{V_1} V_2 = \langle Q_\varepsilon V_1, V_2\rangle Z - \frac{1}{\varepsilon}\alpha(V_2) Q_\varepsilon V_1 - \frac{1}{2\varepsilon}\alpha(V_1) J V_2$, which gives us

$$(\nabla_{V_3}\eta^\varepsilon)_{V_1} V_2 = -\varepsilon\left\langle (\nabla_{V_3}\tau)V_1, V_2\right\rangle Z + \alpha(V_2)(\nabla_{V_3}\tau)V_1,$$

$$\eta^{\varepsilon}_{T(V_1,V_3)} V_2 = \alpha(V_1)\,\langle Q_\varepsilon \tau V_3, V_2\rangle\, Z - \frac{1}{\varepsilon}\alpha(V_1)\alpha(V_2) Q_\varepsilon \tau V_3 \\ - \alpha(V_3)\,\langle Q_\varepsilon \tau V_1, V_2\rangle\, Z + \frac{1}{\varepsilon}\alpha(V_3)\alpha(V_2) Q_\varepsilon \tau V_1 + \frac{1}{2\varepsilon}\langle J V_1, V_3\rangle J V_2$$

Using these identities, we have

$$\begin{aligned} &\langle R^{\varepsilon}(Y, JY)JY, Y\rangle_\varepsilon - \langle R(Y, JY)JY, Y\rangle \\ &= -\|Y\|^2 \langle \eta^{\varepsilon}_Z JY, Y\rangle + \langle [\eta^{\varepsilon}_Y, \eta^{\varepsilon}_{JY}] JY, Y\rangle \\ &= -\frac{1}{2\varepsilon}\|Y\|^4 - \frac{1}{\varepsilon}\langle Q_\varepsilon JY, JY\rangle \langle Q_\varepsilon Y, Y\rangle + \frac{1}{\varepsilon}\langle Q_\varepsilon Y, JY\rangle\langle Q_\varepsilon JY, Y\rangle \\ &= -\frac{1}{2\varepsilon}\|Y\|^4 + \varepsilon\langle \tau Y, Y\rangle^2 + \varepsilon\langle \tau Y, JY\rangle^2 - \frac{1}{4\varepsilon}\|Y\|^4 \\ &= -\frac{3}{4\varepsilon}\|Y\|^4 + \varepsilon\|\tau\|^2\|Y\|^4. \end{aligned}$$

The connection ∇ preserves E and $E^{\perp}$, and hence the same holds for its curvature operator. As a result,

$$\begin{aligned} \langle R^{\varepsilon}(Y, JY)Y, Z\rangle_\varepsilon &= \frac{1}{\varepsilon}\langle (\nabla_Y \eta^{\varepsilon})_{JY} Y - (\nabla_{JY}\eta^{\varepsilon})_Y Y - \eta^{\varepsilon}_Z Y + [\eta^{\varepsilon}_Y, \eta^{\varepsilon}_{JY}] Y, Z\rangle \\ &= -\langle (\nabla_Y \tau) JY - (\nabla_{JY}\tau) Y, Y\rangle. \end{aligned}$$

$$\begin{aligned} \langle R^{\varepsilon}(Y, Z)Z, Y\rangle_\varepsilon &= \langle (\nabla_Y \eta^{\varepsilon})_Z Z - (\nabla_Z \eta^{\varepsilon})_Y Z - \eta^{\varepsilon}_{\tau Y} Z + [\eta^{\varepsilon}_Y, \eta^{\varepsilon}_Z] Z, Y\rangle \\ &= \langle -(\nabla_Z \tau) Y + \frac{1}{\varepsilon} Q_\varepsilon \tau Y - \frac{1}{2\varepsilon^2} J Q_\varepsilon Y, Y\rangle \\ &= -\langle (\nabla_Z \tau) Y, Y\rangle - \frac{1}{\varepsilon}\langle \tau Y, JY\rangle - \|\tau Y\|^2 + \frac{1}{2\varepsilon^2}\|Y\|^2. \\ \langle R^{\varepsilon}(Y, Z)JY, Z\rangle_\varepsilon &= -\langle -(\nabla_Z \tau) Y + \frac{1}{\varepsilon} Q_\varepsilon \tau Y - \frac{1}{2\varepsilon^2} J Q_\varepsilon Y, JY\rangle \\ &= \langle (\nabla_Z \tau) Y, JY\rangle - \frac{1}{\varepsilon}\langle \tau Y, Y\rangle + \langle \tau Y, \tau JY\rangle. \end{aligned}$$

This completes the proof. □

4 Embedded surfaces and variational metric

In this section, it will be convenient to introduce notation

$$b_\varepsilon = \sqrt{1 - (1-\varepsilon)a^2},$$

such that $\frac{1}{b_\varepsilon}$ is uniformly bounded for $0 < \varepsilon < 1$, while unbounded for $\varepsilon = 0$. For our surface $\Sigma \subseteq M$, we write $h^{\varepsilon} = g^{\varepsilon}|_{T\Sigma}$ for the induced Riemannian metric from g^ε and we write σ^ε for the corresponding volume form. Recall the definition of Σ' from (2.3) with h-orthonormal basis X, X_2. The tangent bundle $T\Sigma'$ will have a positively oriented, orthonormal basis X, $\hat{X}_2^\varepsilon$ with respect to h^ε, with

$$\hat{X}_2^\varepsilon = \tfrac{\sqrt{\varepsilon}}{b_\varepsilon} X_2, \quad \text{giving us} \quad \sigma^\varepsilon = \tfrac{b_\varepsilon}{\sqrt{\varepsilon}} \sigma.$$

Using the orientation of Σ and M, we also have the unit normal vector field,

$$\hat{N}^\varepsilon = \tfrac{1}{b_\varepsilon} N^\varepsilon \qquad N^\varepsilon = \varepsilon a Z - \sqrt{1-a^2} JX.$$

We denote corresponding scalar-valued second fundamental form by II^ε.

We want to consider the Gaussian curvature K^ε of h^ε. It will be sufficient to find the formula on Σ'. If R^ε is the curvature operator of ∇^ε then

$$\begin{aligned} K^\varepsilon &= \langle R^\varepsilon(X, X_2^\varepsilon) X_2^\varepsilon, X\rangle_\varepsilon + \mathrm{II}^\varepsilon(X,X)\,\mathrm{II}^\varepsilon(X_2^\varepsilon, X_2^\varepsilon) - \mathrm{II}^\varepsilon(X, X_2^\varepsilon)^2, \\ &=: \mathrm{Sec}^\varepsilon + \mathrm{II}_{11}^\varepsilon\,\mathrm{II}_{22}^\varepsilon - (\mathrm{II}_{12}^\varepsilon)^2. \end{aligned} \tag{4.1}$$

We will use this Gauss Equation to compute the curvature.

Lemma 4.1 *On Σ', we have*

$$\mathrm{II}_{11}^\varepsilon = -\tfrac{1}{b_\varepsilon}\left(\varepsilon a \tau_0 + b_0 \langle \nabla_X X, JX\rangle\right), \qquad \mathrm{II}_{22}^\varepsilon = \tfrac{\varepsilon}{b_\varepsilon}\left(-\tfrac{X_2 a}{b_\varepsilon^2 b_0} + a\tau_0\right),$$

$$\mathrm{II}_{12}^\varepsilon = -\frac{\sqrt{\varepsilon} X a}{b_0 b_\varepsilon^2} + \frac{1-2\varepsilon\tau_1}{2\sqrt{\varepsilon}} = -\frac{\sqrt{\varepsilon}}{b_\varepsilon^2}\left(b_0\langle \nabla_{X_2} X, JX\rangle + a^2(1+\varepsilon\tau_1)\right) + \frac{1}{2\sqrt{\varepsilon}}.$$

Proof If we write $X = X_1$, then for $i, j = 1, 2$,

$$\begin{aligned} \mathrm{II}^\varepsilon(X_i, X_j) &= -\langle \nabla_{X_i}^\varepsilon \hat{N}_\varepsilon, X_j\rangle_\varepsilon = -\frac{1}{b_\varepsilon}\langle \nabla_{X_i}^\varepsilon N_\varepsilon, X_j\rangle_\varepsilon, \\ &= -\tfrac{X_i a}{b_\varepsilon}\left\langle \varepsilon Z + \tfrac{a}{b_0} JX, X_j\right\rangle_\varepsilon - \tfrac{\varepsilon a}{b_\varepsilon}\langle \nabla_{X_i}^\varepsilon Z, X_j\rangle_\varepsilon + \tfrac{b_0}{b_\varepsilon}\langle \nabla_{X_i}^\varepsilon JX, X_j\rangle_\varepsilon. \end{aligned}$$

We see that

$$\begin{aligned} \nabla_X^\varepsilon Z &= \tfrac{1}{2\varepsilon}\left(-JX + 2\varepsilon\tau_0 X + 2\varepsilon\tau_1 JX\right), \\ \nabla_{X_2}^\varepsilon Z &= \tfrac{a}{2\varepsilon}\left(X - 2\varepsilon\tau_0 JX + 2\varepsilon\tau_1 X\right), \\ \nabla_X^\varepsilon JX &= J\nabla_X X + \tfrac{1}{2}\left(1 - 2\varepsilon\tau_1\right) Z, \\ \nabla_{X_2}^\varepsilon JX &= J\nabla_{X_2} X + a\varepsilon\tau_0 Z + \tfrac{1}{2\varepsilon} b_0 X. \end{aligned}$$

Combining formulas, we get the result. □

An important observation from this lemma is that the expressions $\mathrm{II}_{11}^\varepsilon$, $\mathrm{II}_{22}^\varepsilon$ and $\mathrm{II}_{12}^\varepsilon$ are all uniformly bounded on Σ' for fixed $\varepsilon > 0$. The same hold for terms τ_0 and τ_1. This observation is a consequence of the fact that τ and II^ε are tensors on the compact

Σ which have to be uniformly bounded relative to g^ε. We obtain the next important corollary.

Corollary 4.2 *The following expressions are uniformly bounded on* Σ',

$$\frac{Xa}{b_0}, \qquad \frac{X_2 a}{b_0}, \qquad b_0\langle \nabla_X X, JX\rangle, \qquad b_0\langle \nabla_{X_2} X, JX\rangle.$$

Remark 4.3 Observe that (2.4) can be reproduced from the formula of $\mathrm{II}^\varepsilon_{12}$. We note that (2.4) can we rewritten as

$$\begin{aligned}
\frac{Xa}{\sqrt{1-a^2}} = \frac{Xa}{b_0} &= a^2 + \frac{b_0}{a}\langle \nabla^h_{X_2} X, X_2\rangle \\
&\quad + \frac{b_0}{a}\left\langle -\langle \frac{1}{2}JX_2 - \tau X_2, X\rangle Z + \frac{1}{2} b_0 JX, X_2\right\rangle - b_0^2\tau_1 \\
&= a^2 + \frac{b_0}{a}\operatorname{div}(X) + \frac{b_0}{a}\left(ab_0(\frac{1}{2} + \tau_1) + \frac{1}{2}b_0 a\right) - b_0\tau_1 \\
&= 1 + \frac{b_0}{a}\operatorname{div}(X),
\end{aligned}$$

reproducing the observation for the divergence of X in Remark 2.4.

Remark 4.4 Considering the foliation along X, we look at its geodesic curvature of the leaves relative to h^ε,

$$k^\varepsilon_E = \langle \nabla^\varepsilon_X X, \hat{X}^\varepsilon_2\rangle_\varepsilon = -\langle X, \nabla^\varepsilon_X \hat{X}^\varepsilon_2\rangle_\varepsilon = -\operatorname{div}_\varepsilon(\hat{X}^\varepsilon_2)$$

where the divergence $\operatorname{div}_\varepsilon$ is with respect to σ^ε. We can then use

$$\operatorname{div}_\varepsilon(\hat{X}^\varepsilon_2)\sigma^\varepsilon = d\iota_{\hat{X}^\varepsilon_2}\sigma^\varepsilon = d\iota_{X_2}\sigma = \operatorname{div}(X_2)\sigma = \frac{\sqrt{\varepsilon}}{b_\varepsilon}\operatorname{div}(X_2)\sigma^\varepsilon.$$

We consider the limit $k^0_{\Sigma,E} = \frac{1}{\sqrt{\varepsilon}}k^\varepsilon_{\Sigma,E} = \frac{1}{\sqrt{1-a^2}}k^1_E = -\frac{1}{\sqrt{1-a^2}}\operatorname{div}(X_2)$ as a description of the rate at which the foliation flattens out.

5 Proof of the Gauss-Bonnet formula

5.1 The primitive of the curvature integral

Let X be the orthonormal basis of $E \cap T\Sigma'$ as defined in Section 2.2, with $X_2 = b_0 Z + aJX$. Recall definitions of τ_0 and τ_1. Also recall that K^ε is the Gaussian curvature of (Σ, h^ε), while Sec^ε is the sectional curvature of $T\Sigma$ in (M, g^ε) as defined in (4.1). We isolate the order for the different terms that will appear in our integral $\int_\Sigma K^\varepsilon\sigma^\varepsilon$ for the Gauss-Bonnet theorem.

Lemma 5.1 *We have expansion in terms of ε as*

$$K^{\varepsilon}\sigma^{\varepsilon} = \left(\frac{\sqrt{\varepsilon}}{b_{\varepsilon}}\left(\frac{1}{\varepsilon}B_{1,-1} + B_{1,0}\right) + \left(\frac{\sqrt{\varepsilon}}{b_{\varepsilon}}\right)^3\left(\frac{1}{\varepsilon}B_{2,-1} + B_{2,0}\right)\right)\sigma \tag{5.1}$$

with uniformly bounded terms

$$B_{1,-1} = \frac{Xa}{b_0} - a^2, \qquad B_{2,-1} = \langle \nabla_X X, JX\rangle X_2 a - \frac{(Xa)^2}{b_0^2}, \qquad B_{2,0} = a\tau_0\frac{X_2 a}{b_0},$$

$$B_{1,0} = \mathrm{Sec}^1 + a^2\left(1 - \tau_0^2\right) - \left(\tau_1 - \frac{1}{2}\right)^2 - b_0 a\tau_0\langle \nabla_X X, JX\rangle - 2\tau_1\frac{Xa}{b_0}.$$

Proof We consider the Gaussian curvature using the Gauss equation (4.1). We first have the expansion

$$\begin{aligned}
\frac{b_{\varepsilon}^2}{\varepsilon}\mathrm{Sec}^{\varepsilon} &= \langle R^{\varepsilon}(X, X_2)X_2, X\rangle_{\varepsilon} \\
&= a^2\langle R^{\varepsilon}(X, JX)JX, X\rangle_{\varepsilon} + b_0^2\langle R^{\varepsilon}(X, Z)Z, X\rangle_{\varepsilon} - 2ab_0\langle R^{\varepsilon}(X, JX)X, Z\rangle_{\varepsilon} \\
&= a^2\left(\langle R^1(X, JX)JX, X\rangle - \frac{3}{4\varepsilon} + \varepsilon(\tau_0^2 + \tau_1^2)\right) \\
&\quad + b_0^2\left(\frac{1}{4\varepsilon^2} - \tau_0^2 - \tau_1^2 - \langle(\nabla_Z\tau)X, X\rangle - \frac{1}{\varepsilon}\tau_1\right) \\
&\quad - 2ab_0\left(\langle(\nabla_{JX}\tau)X, X\rangle - \langle(\nabla_X\tau)JX, X\rangle\right) \\
&= \mathrm{Sec}^1 + a^2\left(\frac{3}{4} - (\tau_0^2 + \tau_1^2)\right) - \frac{b_0^2}{4} + b_0^2\tau_1 + a^2\left(-\frac{3}{4\varepsilon} + \varepsilon(\tau_0^2 + \tau_1^2)\right) \\
&\quad + \frac{b_0^2}{4\varepsilon^2} - \frac{b_0^2}{\varepsilon}\tau_1 \\
&= \mathrm{Sec}^1 + a^2\left(-\frac{1}{\varepsilon} + \frac{3}{4} - (\tau_0^2 + \tau_1^2) + \tau_1 + \varepsilon\left(\frac{1}{4} + \tau_0^2 + \tau_1^2 - \tau_1\right)\right) \\
&\quad + b_{\varepsilon}^2\left(\frac{1}{4\varepsilon^2} - \frac{1}{\varepsilon}\tau_1 - \frac{1}{4} + \tau_1\right).
\end{aligned}$$

Writing out the identity $K^{\varepsilon}\sigma^{\varepsilon} = \frac{b_{\varepsilon}}{\sqrt{\varepsilon}}(\mathrm{Sec}^{\varepsilon} + \mathrm{II}_{11}^{\varepsilon}\,\mathrm{II}_{22}^{\varepsilon} - (\mathrm{II}_{12}^{\varepsilon})^2)\sigma$, and using Lemma 4.1,

$$\begin{aligned}
\frac{b_{\varepsilon}}{\sqrt{\varepsilon}}\mathrm{Sec}^{\varepsilon} = \frac{\sqrt{\varepsilon}}{b_{\varepsilon}}\Bigg(\mathrm{Sec}^1 + a^2\bigg(-\frac{1}{\varepsilon} + \frac{3}{4} - (\tau_0^2 + \tau_1^2) + \tau_1 \\
+\varepsilon\left(\frac{1}{4} + \tau_0^2 + \tau_1^2 - \tau_1\right)\bigg)\Bigg) \\
+ \frac{b_{\varepsilon}}{\sqrt{\varepsilon}}\left(\frac{1}{4\varepsilon} - \tau_1 - \varepsilon\left(\frac{1}{4} - \tau_1\right)\right),
\end{aligned}$$

$$\frac{b_\varepsilon}{\sqrt{\varepsilon}} \mathrm{II}^\varepsilon_{11}\, \mathrm{II}^\varepsilon_{22} = -\frac{\sqrt{\varepsilon}}{b_\varepsilon}\left(\varepsilon a^2 \tau_0^2 + b_0 a \tau_0 \langle \nabla_X X, JX\rangle\right) + \frac{\sqrt{\varepsilon}}{b_\varepsilon^3}\left(\varepsilon a \tau_0 \frac{X_2 a}{b_0} + \langle \nabla_X X, JX\rangle X_2 a\right),$$

$$-\frac{b_\varepsilon}{\sqrt{\varepsilon}}(\mathrm{II}^\varepsilon_{12})^2 = -\frac{b_\varepsilon}{\sqrt{\varepsilon}}\left(\frac{\varepsilon (Xa)^2}{b_0^2 b_\varepsilon^4} + \frac{1-4\varepsilon\tau_1+4\varepsilon^2\tau_1^2}{4\varepsilon} - \frac{Xa(1-2\varepsilon\tau_1)}{b_0 b_\varepsilon^2}\right)$$
$$= -\frac{\sqrt{\varepsilon}}{b_\varepsilon^3}\frac{(Xa)^2}{b_0^2} + \frac{\sqrt{\varepsilon}}{b_\varepsilon}\frac{Xa(1-2\varepsilon\tau_1)}{\varepsilon b_0} - \frac{b_\varepsilon}{\sqrt{\varepsilon}}\frac{1-4\varepsilon\tau_1+4\varepsilon^2\tau_1^2}{4\varepsilon}.$$

Adding all of these terms together, we have

$$\frac{\sqrt{\varepsilon}}{b_\varepsilon}\left(\mathrm{Sec}^1 + a^2\left(1-\tau_0^2\right) - a^2\left(\tau_1 - \frac{1}{2}\right)^2 - b_0 a \tau_0 \langle \nabla_X X, JX\rangle - 2\tau_1 \frac{Xa}{b_0}\right)$$
$$+ \frac{\sqrt{\varepsilon}}{b_\varepsilon}\left(\frac{Xa}{\varepsilon b_0} - \frac{a^2}{\varepsilon} + a^2\varepsilon\left(\tau_1 - \frac{1}{2}\right)^2\right) - b_\varepsilon\sqrt{\varepsilon}\left(\tau_1 - \frac{1}{2}\right)^2$$
$$+ \frac{\sqrt{\varepsilon}}{b_\varepsilon^3}\left(\varepsilon a \tau_0 \frac{X_2 a}{b_0} + \langle \nabla_X X, JX\rangle X_2 a - \frac{(Xa)^2}{b_0^2}\right)$$

which equals our result. The terms are uniformly bounded by Corollary 4.2. □

5.2 Proof of Theorem 1.1

Let us first state the following result related to limits of integrals that are singular at the limit.

Lemma 5.2 *Let $f : [-1, 1] \to \mathbb{R}$ be a continuous, uniformly bounded function and write $b_\varepsilon = \sqrt{1-(1-\varepsilon)a^2}$.*

(a) *The following limits of integrals vanish;*

$$\lim_{\varepsilon\to 0} \sqrt{\varepsilon} \int_{-1}^{1} \frac{f(a)}{b_\varepsilon} da = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \varepsilon \int_{-1}^{1} \frac{f(a)}{b_\varepsilon^3} da = 0.$$

(b) *We have $\lim_{\varepsilon\to 0} \int_{-1}^{1} \frac{f(a)}{b_\varepsilon} da = \int_{-1}^{1} \frac{f(a)}{b_0} da$ with the latter integral being well-defined and finite.*
(c) *The limit $\lim_{\varepsilon\to 0} \sqrt{\varepsilon} \int_{-1}^{1} \frac{f(a)}{b_\varepsilon^3} da$ exits and equals*

$$\lim_{\varepsilon\to 0} \sqrt{\varepsilon} \int_{-1}^{1} \frac{f(a)}{b_\varepsilon^3} da = f(-1) + f(1).$$

(d) *The limit* $\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\varepsilon}}\int_{-1}^{1} f(a)(\frac{1}{b_\varepsilon}-\frac{1}{b_0})da$ *exists and equals*

$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\varepsilon}}\int_{-1}^{1} f(a)\left(\frac{1}{b_\varepsilon}-\frac{1}{b_0}\right)da = -f(1)-f(-1).$$

Proof The statements for the integral $\int_{-1}^{1}\frac{f(a)}{b_\varepsilon}da$ follow from the identity

$$\int_{-1}^{1}\frac{f(a)}{\sqrt{1+(\varepsilon-1)a^2}}da = \sqrt{1-\varepsilon}\int_{-\frac{\sin^{-1}\sqrt{1-\varepsilon}}{\sqrt{1-\varepsilon}}}^{\frac{\sin^{-1}\sqrt{1-\varepsilon}}{\sqrt{1-\varepsilon}}} f(\tfrac{1}{\sqrt{1-\varepsilon}}\sin(u))du,$$

which converge to $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\sin(u))du$. We similarly have

$$\int_{-1}^{1}\frac{f(a)}{(1+(\varepsilon-1)a^2)^{3/2}}da = \int_{-\varepsilon^{-1/2}}^{\varepsilon^{1/2}} f\left(\frac{u}{\sqrt{(1-\varepsilon)u^2+1}}\right)du.$$

Obviously, this integral has upper bound for any $0<k<\frac{1}{\sqrt{\varepsilon}}$,

$$\begin{aligned}
&\int_{-k}^{k} f\left(\frac{u}{\sqrt{1-(1-\varepsilon)u^2}}\right)du + \left(\frac{1}{\sqrt{\varepsilon}}-k\right)\\
&\left(\max_{-1\le a\le -\frac{k}{\sqrt{(1-\varepsilon)k^2+1}}} f(a) + \max_{\frac{k}{\sqrt{(1-\varepsilon)k^2+1}}\le a\le 1} f(a)\right)\\
&\le \int_{-k}^{k} f\left(\frac{u}{\sqrt{1-(1-\varepsilon)u^2}}\right)du + \left(\frac{1}{\sqrt{\varepsilon}}-k\right)\\
&\left(\max_{-1\le a\le -\frac{k}{\sqrt{k^2+1}}} f(a) + \max_{\frac{k}{\sqrt{k^2+1}}\le a\le 1} f(a)\right)
\end{aligned}$$

and an analogous lower bound involving the minimum. As the limit of both of these bounds vanish when multiplied with ε and letting $\varepsilon\to 0$, we have the second part of (a). Multiplying with $\sqrt{\varepsilon}$ and taking the limit, we obtain

$$\begin{aligned}
&\min_{-1\le a\le -\frac{k}{\sqrt{k^2+1}}} f(a) + \min_{\frac{k}{\sqrt{k^2+1}}\le a\le 1} f(a)\\
&\le \lim_{\varepsilon\to 0}\sqrt{\varepsilon}\int_{-1}^{1}\frac{f(a)}{b_\varepsilon^3}da \le \max_{-1\le a\le -\frac{k}{\sqrt{k^2+1}}} f(a) + \max_{\frac{k}{\sqrt{k^2+1}}\le a\le 1} f(a)
\end{aligned}$$

and letting $k\to\infty$, we obtain the result of (c).

Finally, for the result in (d), define for $-1 \leq c_1 < c_2 \leq 1$, the integral $I_\varepsilon(c_1, c_2) = \int_{c_1}^{c_2} \left(\frac{1}{b_\varepsilon} - \frac{1}{b_0}\right) da = F_\varepsilon(c_2) - F_\varepsilon(c_1)$, where $F_\varepsilon(0) = 0$ and $F'_\varepsilon(0) = \frac{1}{b_\varepsilon} - \frac{1}{b_0}$. Then

$$F_\varepsilon(c) = \frac{\sin^{-1}(\sqrt{1-\varepsilon}c)}{\sqrt{1-\varepsilon}} - \sin^{-1}(c),$$

and so

$$\lim_{\varepsilon\to 0} \frac{1}{\sqrt{\varepsilon}} F_\varepsilon(c) = \frac{\partial}{\partial\sqrt{\varepsilon}} \frac{\sin^{-1}(\sqrt{1-\varepsilon}c)}{\sqrt{1-\varepsilon}}\bigg|_{\varepsilon=0} = \begin{cases} -1 & \text{if } c = 1, \\ 1 & \text{if } c = -1, \\ 0 & \text{otherwise.} \end{cases} .$$

Hence, for $-1 < c_1 < c_2 < 1$, we have

$$\lim_{\varepsilon\to 0} \frac{I_\varepsilon(1, c_2)}{\sqrt{\varepsilon}} = -1, \qquad \lim_{\varepsilon\to 0} \frac{I_\varepsilon(c_1, c_2)}{\sqrt{\varepsilon}} = 0, \qquad \lim_{\varepsilon\to 0} \frac{I_\varepsilon(c_1, 1)}{\sqrt{\varepsilon}} = -1.$$

Next, since for any subinterval,

$$\begin{aligned}\left(\min_{c_1\leq a\leq c_2} f(a)\right) I_\varepsilon(c_1, c_2) &\leq \int_{1-\rho}^{1} f(a) \left(\frac{1}{b_\varepsilon} - \frac{1}{b_0}\right) da \\ &\leq \left(\max_{c_1\leq a\leq c_2} f(a)\right) I_\varepsilon(c_1, c_2),\end{aligned}$$

it follows that

$$\lim_{\varepsilon\to 0} \frac{1}{\sqrt{\varepsilon}} \int_{1-\rho}^{1} f(a) \left(\frac{1}{b_\varepsilon} - \frac{1}{b_0}\right) da = -f(1) - f(-1).$$

□

Proof of Theorem 1.1 From the equation (5.1), define a function $A_{ij} : [-1, 1] \to \mathbb{R}$, such that $A_{i,j}(c) = \int_{0\leq a\leq c} B_{i,j}\sigma$ for $c \geq 0$, and $A_{i,j}(c) = -\int_{c\leq a\leq 0} B_{i,j}\sigma$ for c negative. Recall that each B_{ij} is uniformly bounded on Σ by Corollary 4.2, so from Assumption (A), each $A_{i,j}$ is differentiable when $1 - |c| \in (0, \delta)$ for some $\delta < 1$ with a bounded derivative. We look at limits of $\int_\Sigma K^\varepsilon \sigma^\varepsilon = 2\pi\chi(\Sigma)$.

Computations of terms of order $-1/2$. Since $\lim_{\varepsilon\to 0} \sqrt{\varepsilon} \int_\Sigma K^\varepsilon \sigma^\varepsilon = 0$, then

$$\lim_{\varepsilon\to 0} \sqrt{\varepsilon} \int_{|a|<c_1} K^\varepsilon \sigma^\varepsilon = \int_{|a|<c_1} \frac{1}{b_0} B_{1,-1}\sigma$$

and from Lemma 5.2 and its proof

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\int_{c_1\le a\le 1} K^\varepsilon\sigma^\varepsilon \\
&\quad = \lim_{\varepsilon\to 0}\int_{c_1\le a\le 1}\frac{\varepsilon}{b_\varepsilon}\left(\frac{1}{\varepsilon}A'_{1,-1}+A'_{1,0}\right)da - \lim_{\varepsilon\to 0}\int_{c_1\le a\le 1}\frac{\varepsilon}{b_\varepsilon^3}\left(A'_{2,0}+A'_{2,1}\varepsilon\right)da \\
&\quad = \int_{c_1\le a\le 1}\frac{1}{b_0}A'_{1,-1}da = \int_{c_1\le a\le 1}\frac{B_{1,-1}}{b_0}\sigma.
\end{aligned}
$$

and similarly $\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\int_{-1\le a\le -c_1} K^\varepsilon\sigma^\varepsilon = \int_{c_1\le a\le 1}\frac{B_{1,-1}}{b_0}\sigma$. In conclusion,

$$
\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\int_\Sigma K^\varepsilon\sigma^\varepsilon = \int_\Sigma \frac{B_{1,-1}}{b_0}\sigma = 0.
$$

Computations of terms of order 0. Using our computations for order $-1/2$, we have that

$$
\begin{aligned}
\int_\Sigma K^\varepsilon\sigma^\varepsilon &= \int_\Sigma K^\varepsilon\sigma^\varepsilon - \int_\Sigma \frac{B_{-1,1}}{b_0}\sigma \\
&= \int_\Sigma\left(\left(\frac{1}{\sqrt{\varepsilon}}B_{1,-1}\left(\frac{1}{b_\varepsilon}-\frac{1}{b_0}\right)+\frac{\sqrt{\varepsilon}}{b_\varepsilon}B_{1,0}\right)+\frac{\sqrt{\varepsilon}}{b_\varepsilon^3}(B_{2,-1}+B_{2,0}\varepsilon)\right)\sigma \\
&= \int_{-1}^{1}\left(\left(\frac{1}{\sqrt{\varepsilon}}A'_{1,-1}\left(\frac{1}{b_\varepsilon}-\frac{1}{b_0}\right)+\frac{\sqrt{\varepsilon}}{b_\varepsilon}A'_{1,0}\right)+\frac{\sqrt{\varepsilon}}{b_\varepsilon^3}(A'_{2,-1}+A'_{2,0}\varepsilon)\right)da
\end{aligned}
$$

and using Lemma 5.2, we finally have

$$
\lim_{\varepsilon\to 0}\int_\Sigma K^\varepsilon\sigma^\varepsilon = 2\pi\chi(\Sigma) = A'_{2,-1}(1) - A'_{1,-1}(1) + A'_{2,-1}(-1) - A'_{1,-1}(-1).
$$

For the final part,

$$
\begin{aligned}
B_{1,-1} &= \frac{Xa}{b_0} - a^2 = 1 - a^2 + \frac{b_0}{a}\operatorname{div}(X), \\
B_{2,-1} &= \langle \nabla_X X, JX\rangle X_2 a - \frac{(Xa)^2}{b_0^2} = -\langle X, \nabla_X X_2\rangle\frac{X_2 a}{a} - \frac{(Xa)^2}{b_0^2} \\
&= -\langle X, \nabla_X^h X_2\rangle\frac{X_2 a}{a} + \langle X, b_0\tau X\rangle\frac{X_2 a}{a} - \frac{(Xa)^2}{b_0^2} \\
&= (\operatorname{div}(X_2) - b_0\tau_0)\,\frac{X_2 a}{a} - \left(1 + \frac{\sqrt{1-a^2}}{a}\operatorname{div}(X)\right)^2
\end{aligned}
$$

Using Remark 2.4 and Remark 4.4 with $a = \cos\vartheta$, we can also write

$$
B_{1,-1} = -X\vartheta - \cos^2\vartheta, \qquad B_{2,-1} = \frac{\sin^2\vartheta}{\cos\vartheta}(k_E^0 + \tau_0)X_2\vartheta - (X\vartheta)^2
$$

We notice that $K_{\Sigma,E} = B_{2,-1} - B_{1,-1}$ for the result. □

Remark 5.3 We observe as a corollary of the proof of Theorem 1.1, since the terms of order $-\frac{1}{2}$ in ε has to vanish and we will have

$$\int_\Sigma \frac{B_{1,-1}}{b_0}\sigma = \int_\Sigma \left(\frac{1}{1-a^2} - \frac{a^2}{\sqrt{1-a^2}}\right)\sigma = \int_\Sigma \left(\sqrt{1-a^2} - \frac{1}{a}\operatorname{div}(X)\right)\sigma = 0.$$

6 Analysis of surfaces with boundary

For the final section, we will consider manifolds with boundary. Let (M, E, g) be a sub-Riemannian contact manifold. We will assume that we have a compact C^2-surface $\Sigma \subset M$ with a boundary $\partial\Sigma$ that is piecewise C^2. Let h be the induced from the metric $g = g_1$. We parametrize $\partial\Sigma$ by a piecewise C^2 curve $\gamma : [0, \ell] \to \partial\Sigma$, parametrized by h-arc length and positively oriented. Relative to the restriction of the contact form $\beta = \alpha|_{T\Sigma}$, define

$$W = \{t \in [0, \ell] \, : \, \beta(\dot\gamma(t)) \neq 0\}.$$

- We say that $t_1 \in W^+$ if t_1 is a left limit point of W such that $\beta(\dot\gamma(t_1+)) = 0$.
- We say that $t_2 \in W^-$ if t_2 is a right limit point of W such that $\beta(\dot\gamma(t_2-)) = 0$.

Note that W^+ and W^- are not necessarily disjoint, and that there may be limit points of left or right limit points that are not in $W^\pm$, since they could be corner points. We will assume that the following holds.

$$\begin{gathered}\text{If } t_1 \in W^+ \text{ (resp. } t_2 \in W^-\text{) then} \\ \tfrac{d}{dt}\beta(\dot\gamma)(t_1+) \neq 0 \text{ (resp. } \tfrac{d}{dt}\beta(\dot\gamma)(t_2-) \neq 0).\end{gathered} \tag{B}$$

The assumption (B) has the following geometric interpretation: Any transition point where the boundary goes from not being tangent to E to being tangent to E, has to either be an isolated point tangent to E or it must happen at a point where the boundary fails to be C^2, see Remark 6.3 for more information. Let X and X_2 be as in Section 2.2. For $\gamma(t) \notin \operatorname{char}(\Sigma)$, write

$$\dot\gamma(t) = \cos\theta(t)X + \sin\theta(t)X_2. \tag{6.1}$$

Let $S = \{y_1, \dots, y_N\}$ be the set of points where $\partial\Sigma$ fails to be C^2, each with exterior angles $\phi_1^\varepsilon, \dots, \phi_N^\varepsilon$ with respect to h^ε. Write $\phi_j^1 = \phi_j$. We define $S = S_2 \cup S_1 \cup S_0$, where S_n contains the points $y_i = \gamma(c)$ satisfying that precisely n of the vectors $\dot\gamma(c-)$ and $\dot\gamma(c+)$ are in $E_{y_i} \cap T_{y_i}\Sigma$. Define k_g^ε as the signed geodesic curvature of $\partial\Sigma$ with respect to h^ε with $k_g^1 = k_g$.

Theorem 6.1 (Sub-Riemannian Gauss-Bonnet theorem with boundary) *For points $y = \gamma(t) \in \partial\Sigma$ where the below functions make sense, define*

$$p^\pm(y) = \operatorname{sign}(\langle X, \dot\gamma(t\pm)\rangle) \qquad q^\pm(y) = \operatorname{sign}(\langle X_2, \dot\gamma(t\pm)\rangle)$$

Write $\hat{W}_t^{\pm} = \{t \in W^{\pm} \,:\, \gamma(t) \notin \mathrm{char}(\Sigma)\}$, *and define*

$$w(t) = \cos\theta(t) \frac{\frac{d}{dt}\beta(\dot{\gamma})(t) - a(\gamma(t))\cos\theta(t)\sin\theta(t)}{|\frac{d}{dt}\beta(\dot{\gamma})(t)|}.$$

If (A) *and* (B) *hold, then*

$$\begin{aligned}
2\pi\chi(\Sigma) &= \frac{d}{dc}\bigg|_{c=0} \int_{|a|\geq 1-c} K_{\Sigma,E}\sigma + \int_{\partial\Sigma\cap\mathrm{char}(\Sigma)} k_g(s)ds + \sum_{y_i\in S_2} \phi_i \\
&\quad + \frac{\pi}{2}\sum_{y_i\in S_1} \mathrm{sign}(\phi_i) + \sum_{y_i\in S_0} \frac{\pi}{2}(1 - q^+(y_i)q^-(y_i))\,\mathrm{sign}(\phi_i) \\
&\quad + \frac{\pi}{2}\sum_{t\in\hat{W}^+} q^+(\gamma(t))p^+(\gamma(t)) + \frac{\pi}{2}\sum_{t\in\hat{W}^-} q^-(\gamma(t))p^-(\gamma(t)) \\
&\quad + \frac{\pi}{2}\sum_{t\in W^+\cap\mathrm{char}(\Sigma^+)} w(t+) - \frac{\pi}{2}\sum_{t\in W^-\cap\mathrm{char}(\Sigma^+)} w(t-)
\end{aligned}$$

We can interpret the result as follows. If the boundary has a segment that is contained in the characteristic set, then its geodesic curvature contribute as usual, as do corners with both vectors in E, i.e., points in S_2. For curves outside the characteristic set, geodesic curvature of paths tangent to E and non-tangent to E approach respectively zero and infinity, and neither will hence contribute to the Gauss-Bonnet formula. If a vector is not in E, it will become more and more orthogonal to E, so, up to sign, we get an angle of $\frac{\pi}{2}$ for every point S_1 as well as for every point where the curve transitions from non-tangent to E to following E outside of the characteristic set. Vectors outside of E also become more parallel to each other, so angles in S_0 approach either 0 or $\pm\pi$.

Proof Our goal will be to show that

$$\int_\Sigma K^\varepsilon d\sigma^\varepsilon + \int_{\partial\Sigma} k_g^\varepsilon(s^\varepsilon)ds^\varepsilon + \sum_{j=1}^N \phi_j^\varepsilon = C_0 + \frac{C_1}{\sqrt{\varepsilon}} + \frac{C_2\log\varepsilon}{\sqrt{\varepsilon}} + o(1),$$

for some constants C_0, C_1, C_2. It then follows from the Gauss-Bonnet theorem that $C_1 = C_2 = 0$ and that $C_0 = 2\pi\chi(\Sigma)$. The conclusion follows from proving that C_0 equals the expression in Theorem 6.1. The proof will proceed in parts. We first compute geodesic curvature along C^2-components of the boundary, then establishing the limiting behavior of the resulting integrals. Finally, there is an analysis of the limiting behavior of corners. □

Geodesic curvature along C^2 components of the boundary

We let $\gamma(t)$ be a parametrization of the boundary $\partial\Sigma$ by h-arc length defined on $[0, \ell]$. Write

$$\gamma_\varepsilon(s) = \gamma(\varphi_\varepsilon(s)), \qquad \frac{d}{ds}\varphi_\varepsilon(s) = \frac{1}{\|\gamma(\varphi_\varepsilon(s))\|_\varepsilon}, \qquad \varphi_\varepsilon(0) = 0.$$

for its reparametrization by $h^\varepsilon = g^\varepsilon|_\Sigma$-arc length defined for $s \in [0, \ell^\varepsilon]$ with $\ell^\varepsilon = \varphi_\varepsilon^{-1}(\ell)$. Let I^ε denote $\frac{\pi}{2}$-rotation on $T\Sigma$ in the positive direction with respect to h^ε. For any $s = s^\varepsilon \in [0, \ell^\varepsilon]$, the h^ε-geodesic curvature of γ_ε at s equals $k_g^\varepsilon(s) = \langle D_s^\varepsilon \dot\gamma_\varepsilon, I^\varepsilon \dot\gamma_\varepsilon\rangle_\varepsilon$ where D_t^ε is the covariation derivative with respect to ∇^ε along the curve $\gamma(t)$. If ds^ε denotes the increment with respect to h^ε-arc length, then we are interested in computing the integral

$$\int_{\partial\Sigma} k_g^\varepsilon(s^\varepsilon) ds^\varepsilon = \int_0^\ell \frac{1}{\|\dot\gamma\|_\varepsilon^2} \langle D_t^\varepsilon \dot\gamma, I^\varepsilon \dot\gamma\rangle_\varepsilon \, dt.$$

We want to find the part of this integral that has order zero with respect to ε.

Decomposition into subintervals

Write $[0, \ell] = T = T_0 \cup T_1 \cup T_2 \cup T_3$ where

$$\begin{aligned}
T_0 &= \{t \in T \,:\, \gamma(t) \in S\},\\
T_1 &= \{t \in T \setminus T_0 \,:\, \gamma(t) \in \mathrm{char}(\Sigma), \dot\gamma(t) \text{ defined}\},\\
T_2 &= \{t \in T \setminus T_0 \,:\, \gamma(t) \notin \mathrm{char}(\Sigma), \dot\gamma(t) \in E_{\gamma(t)}\} = \{t \in T \,:\, \theta = 0\},\\
T_3 &= \{t \in T \setminus T_0 \,:\, \gamma(t) \notin \mathrm{char}(\Sigma), \dot\gamma(t) \notin E_{\gamma(t)}\} = \{t \in T \,:\, \theta \neq 0\}.
\end{aligned}$$

In particular, T_3 is an open subset of T. Since T_0 consists of isolated points,

$$\textstyle\int_{\partial\Sigma} k^\varepsilon(s^\varepsilon) ds^\varepsilon = \int_{(s^\varepsilon)^{-1}(T_1 \cup T_2 \cup T_3)} k^\varepsilon(s^\varepsilon) ds^\varepsilon.$$

Furthermore, on T_1, we have that $\int_{(s^\varepsilon)^{-1}(T_1)} k^\varepsilon(s^\varepsilon) ds^\varepsilon = \int_{(s^\varepsilon)^{-1}(T_1)} k^1(s) ds$. We will thus only consider $T_2 \cup T_3$.

Recall that β is the restriction of the contact form to Σ. Introduce the function $\psi(t) = \beta(\dot\gamma(t)) = b_0(\gamma(t)) \sin\theta(t)$, and note that

$$\|\dot\gamma\|_\varepsilon^2 = \cos^2\theta + \tfrac{b_\varepsilon^2}{\varepsilon} \sin^2\theta = 1 + (1-\varepsilon) \tfrac{\psi^2}{\varepsilon} = \tfrac{\varepsilon + (1-\varepsilon)\psi^2}{\varepsilon}.$$

Since γ is piecewise C^2, it follows that ψ is C^1 on each piecewise component where it is defined. For $t \in T_2 \cup T_3$, we can then write $\dot\gamma(t)$ as in (6.1), meaning that $I^\varepsilon \dot\gamma(t) = \frac{\sqrt{\varepsilon}}{b_\varepsilon} \cos\theta X_2 - \frac{b_\varepsilon}{\sqrt{\varepsilon}} \sin\theta X$. Then

$$\begin{aligned}
\langle D_t^\varepsilon \dot\gamma, I^\varepsilon \dot\gamma\rangle_\varepsilon &= \left\langle D_t \dot\gamma + \langle Q_\varepsilon \dot\gamma, \dot\gamma\rangle Z - \beta(\dot\gamma) \left(\frac{1}{\varepsilon} Q_\varepsilon + \frac{1}{2\varepsilon} J\right) \dot\gamma, \right.\\
&\qquad \left. \frac{\sqrt{\varepsilon}}{b_\varepsilon} \cos\theta X_2 - \frac{b_\varepsilon}{\sqrt{\varepsilon}} \sin\theta X \right\rangle_\varepsilon\\
&= \left\langle D_t \dot\gamma + \psi \left(\tau \dot\gamma - \frac{1}{\varepsilon} J \dot\gamma\right), \frac{a \cos\theta \sqrt{\varepsilon}}{b_\varepsilon} JX - \frac{b_\varepsilon \sin\theta}{\sqrt{\varepsilon}} X \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\cos\theta b_0}{\sqrt{\varepsilon}b_\varepsilon}\left(\langle D_t\dot\gamma, Z\rangle-\varepsilon\langle\tau\dot\gamma,\dot\gamma\rangle\right)\\
&=\frac{\cos\theta}{\sqrt{\varepsilon}b_\varepsilon}(k_{0,0}+\varepsilon k_{1,0})+\frac{b_\varepsilon\sin\theta}{\sqrt{\varepsilon}^3}\left(k_{0,1}+\varepsilon k_{1,1}\right)\\
&=\frac{\cos\theta}{\sqrt{\varepsilon}b_\varepsilon}(k_{0,0}+\varepsilon k_{1,0})+\frac{\psi}{\sqrt{\varepsilon}^3}\left(k_{0,1}+\varepsilon k_{1,1}\right)\\
&\quad+\frac{\sin\theta}{\sqrt{\varepsilon}(b_\varepsilon+b_0)}\left(k_{0,1}+\varepsilon k_{1,1}\right).
\end{aligned}
$$

with

$$
\begin{aligned}
k_{0,0}&=-\psi a\langle J\dot\gamma, JX\rangle+b_0\langle D_t\dot\gamma, Z\rangle=b_0(-a\cos\theta\sin\theta+\dot\psi),\\
k_{1,0}&=a\langle D_t\dot\gamma, JX\rangle+\psi\langle\tau\dot\gamma, JX\rangle-\langle\tau\dot\gamma,\dot\gamma\rangle,\\
k_{0,1}&=\psi\langle J\dot\gamma, X\rangle=-\psi\sin\theta,\\
k_{1,1}&=\langle D_t\dot\gamma, X\rangle+\psi\langle\tau\dot\gamma, X\rangle.
\end{aligned}
$$

For the integral over T_2, we have $\psi=\sin\theta=\dot\theta=0$, $\cos\theta=\pm1$ and $\|\gamma\|_\varepsilon=1$, so our integral becomes

$$
\int_{(s^\varepsilon)^{-1}(T_2)}k^\varepsilon ds^\varepsilon=\sqrt{\varepsilon}\int_{T_2}\frac{k_{1,0}}{b_\varepsilon}dt\to 0.
$$

since $\frac{\sqrt{\varepsilon}}{b_\varepsilon}$ approaches 1 or 0 at endpoints that respectively are or are not in $\mathrm{char}(\Sigma)$. For the remainder of the proof we will compute $\int_{(s^\varepsilon)^{-1}(T_3)}k^\varepsilon ds^\varepsilon$.

Computations over T_3

We will use the following result to calculate the integral over T_3.

Lemma 6.2 *For any C^1-function $f:[0,\ell]\to\mathbb{R}$, we have*

(a)

$$
\int_{T_3}\frac{\sqrt{\varepsilon}f(t)}{\varepsilon+(1-\varepsilon)\psi(t)^2}dt\to\frac{\pi}{2}\sum_{t\in W^+}\frac{f(t)}{|\dot\psi(t)|}-\frac{\pi}{2}\sum_{t\in W^-}\frac{f(t)}{|\dot\psi(t)|}
$$

(b) *Relative to f, there exists constants such that*

$$
\frac{1}{\sqrt{\varepsilon}}\int_{T_3}\frac{f(t)\psi(t)}{\varepsilon+(1-\varepsilon)\psi(t)^2}dt=\frac{\log\varepsilon}{\sqrt{\varepsilon}}C_{1,f}+\frac{1}{\sqrt{\varepsilon}}C_{2,f}+o(1),
$$

so in particular, there are no terms of degree 0 in ε.

(c) *For the geodesic curvature, we have*

$$\begin{aligned}\int_{s^{-1}(T_3)} k^\varepsilon ds^\varepsilon &= \frac{\pi}{2}\sum_{t\in W^+} \operatorname{sign}(\dot\psi(t+))\cos\theta(t+)\\ &- \frac{\pi}{2}\sum_{t\in W^-} \operatorname{sign}(\dot\psi(t-))\cos\theta(t-) - \sum_{t\in W^+} \frac{a(\gamma(t\pm))\cos^2\theta(t\pm)\sin\theta(\pm t)}{|\dot\psi(t\pm)|}\\ &+ \sum_{t\in W^-} \frac{a(\gamma(t\pm))\cos^2\theta(t\pm)\sin\theta(\pm t)}{|\dot\psi(t\pm)|}.\\ &= \frac{\pi}{2}\sum_{t\in \hat W^+} q^+(\gamma(t))p^+(\gamma(t)) + \frac{\pi}{2}\sum_{t\in \hat W^-} q^-(\gamma(t))p^-(\gamma(t))\\ &+ \frac{\pi}{2}\sum_{t\in W^+\cap \operatorname{char}(\Sigma^+)} w(t+) - \frac{\pi}{2}\sum_{t\in W^-\cap \operatorname{char}(\Sigma^+)} w(t-).\end{aligned}$$

Proof In order to find the integral of the geodesic curvature under T_3, we will first need to consider integrals of the form $\int_{T_3} \frac{\sqrt{\varepsilon} f}{\varepsilon+(1-\varepsilon)\psi^2}\, dt$ and $\frac{1}{\sqrt{\varepsilon}}\int_{T_3} \frac{f\psi}{\varepsilon+(1-\varepsilon)\psi^2}\, dt$ with f being C^1. Consider three types of subsets L_0, L_- and L_+ of T_3, with the following properties.

- L_0 is an interval where ψ is bounded away from zero.
- $L_+ = (c_1, c_2)$ is an open interval, where ψ is bounded away from zero on any interval $(c_1 + \rho, c_2)$, $\rho > 0$, but $\psi(c_1+) = 0$. Furthermore, we assume that $\dot\psi$ is bounded away from zero on L_+.
- $L_- = (c_3, c_4)$ is an open interval, where ψ is bounded away from zero on any interval $(c_3, c_4 - \rho)$, $\rho > 0$, but $\psi(c_4-) = 0$. Furthermore, we assume that $\dot\psi$ is bounded away from zero on L_-.

By our assumption (B), we can decompose our set T_3 into a disjoint union intervals of the above type, where we have one interval of the type L_+ (resp. L_-) for every $c \in W^+$ (resp. $c \in W^-$). For an interval of the type L_0, we have

$$\begin{aligned}\int_{L_0} \frac{\sqrt{\varepsilon} f}{\varepsilon + (1-\varepsilon)\psi^2}\, dt &= O(\sqrt{\varepsilon}),\\ \frac{1}{\sqrt{\varepsilon}}\int_{L_0} \frac{f\psi}{\varepsilon + (1-\varepsilon)\psi^2}\, dt &= \frac{1}{\sqrt{\varepsilon}}\int_{L_0} \frac{f}{\psi} dt + O(\sqrt{\varepsilon}),\end{aligned}$$

neither of which have terms of degree zero. We consider the L_+ and L_- for the remainder of the proof.

(a) For $L_+ = (c_1, c_2)$, we will use that for any sufficiently small $\rho > 0$, $(c_1, c_2) \setminus (c_1, c_1 + \rho)$ is an interval of the type L_0. We hence have that

$$\int_{c_1}^{c_2} \frac{\sqrt{\varepsilon} f}{\varepsilon + (1-\varepsilon)\psi^2}\, dt = \int_{c_1}^{c_1+\rho} \frac{\sqrt{\varepsilon} f}{\varepsilon + (1-\varepsilon)\psi^2}\, dt + O(\sqrt{\varepsilon}),$$

and furthermore,

$$\begin{aligned}
&\frac{\left(\inf_{c_1<t\le c_1+\rho}\frac{f(t)}{\dot\psi(t)}\right)}{\sqrt{1-\varepsilon}}\left(\tan^{-1}\sqrt{\frac{1-\varepsilon}{\varepsilon}}\psi(c_1+\rho)\right)\\
&\le\int_{c_1}^{c_1+\rho}\frac{\sqrt{\varepsilon}f}{\varepsilon+(1-\varepsilon)\psi^2}\,dt\\
&\le\frac{\left(\sup_{c_1<t\le c_1+\rho}\frac{f(t)}{\dot\psi(t)}\right)}{\sqrt{1-\varepsilon}}\left(\tan^{-1}\sqrt{\frac{1-\varepsilon}{\varepsilon}}\psi(c_1+\rho)\right).
\end{aligned}$$

Taking a limit as $\varepsilon\to 0$, we have that

$$\begin{aligned}
&(\operatorname{sign}\psi(c_1+\rho))\frac{\pi}{2}\left(\inf_{c_1<t\le c_1+\rho}\frac{f(t)}{\dot\psi(t)}\right)\\
&\le\lim_{\varepsilon\to 0}\int_{c_1}^{c_1+\rho}\frac{\sqrt{\varepsilon}f}{\varepsilon+(1-\varepsilon)\psi^2}\,dt\le(\operatorname{sign}\psi(c_1+\rho))\frac{\pi}{2}\left(\sup_{c_1<t\le c_1+\rho}\frac{f(t)}{\dot\psi(t)}\right).
\end{aligned}$$

As this should be valid for any ρ, we can let $\rho\to 0$ to obtain

$$\begin{aligned}
&\int_{c_1}^{c_2}\frac{\sqrt{\varepsilon}f}{\varepsilon+(1-\varepsilon)\psi^2}\,dt=(\operatorname{sign}\psi(c_1+))\frac{\pi}{2}\frac{f(c_1+)}{\dot\psi(c_1+)}+o(1)\\
&=\frac{\pi}{2}\frac{f(c_1+)}{|\dot\psi(c_1+)|}+o(1).
\end{aligned}$$

If we do similar computations for $L_-=(c_3,c_4)$, we obtain

$$\int_{c_3}^{c_4}\frac{\sqrt{\varepsilon}f}{\varepsilon+(1-\varepsilon)\psi^2}\,dt=-\frac{\pi}{2}\frac{f(c_4-)}{|\dot\psi(c_4-)|}+o(1).$$

(b) For L^+, using integration by parts, we find that

$$\begin{aligned}
&\frac{1}{\sqrt{\varepsilon}}\int_{c_1}^{c_2}\frac{f\psi}{\varepsilon+(1-\varepsilon)\psi^2}\,dt=\frac{1}{\sqrt{\varepsilon}}\int_{c_1}^{c_1+\rho}\frac{f\psi}{\varepsilon+(1-\varepsilon)\psi^2}\,dt\\
&\quad+\frac{1}{\sqrt{\varepsilon}}\int_{c_1+\rho}^{c_2}\frac{f}{\psi}\,dt+O(\sqrt{\varepsilon})\\
&=\frac{\log(\varepsilon+(1-\varepsilon)\psi(c_1+\rho)^2)}{2\sqrt{\varepsilon}(1-\varepsilon)}\frac{f(c_1+\rho)}{\dot\psi(c_1+\rho)}-\frac{\log(\varepsilon)}{2\sqrt{\varepsilon}(1-\varepsilon)}\frac{f(c_1+)}{\dot\psi(c_1+)}\\
&\quad-\frac{1}{2\sqrt{\varepsilon}(1-\varepsilon)}\int_{c_1}^{c_1+\rho}\log(\varepsilon+(1-\varepsilon)\psi^2)\frac{d}{dt}\left(\frac{f}{\dot\psi}\right)\,dt\\
&\quad+\frac{\log|\psi(c_2)|}{\sqrt{\varepsilon}}\frac{f(c_2)}{\dot\psi(c_2)}-\frac{\log|\psi(c_1+\rho)|}{\sqrt{\varepsilon}}\frac{f(c_1+\rho)}{\dot\psi(c_1+\rho)}
\end{aligned}$$

$$-\frac{1}{\sqrt{\varepsilon}}\int_{c_1+\rho}^{c_2}\log|\psi|\frac{d}{dt}\left(\frac{f}{\dot\psi}\right)dt+O(\sqrt{\varepsilon}).$$

Using that

- $\frac{\log(\varepsilon+(1-\varepsilon)\psi(c_1+\rho)^2)}{2\sqrt{\varepsilon}(1-\varepsilon)}\frac{f(c_1+\rho)}{\dot\psi(c_1+\rho)}-\frac{\log|\psi(c_1+\rho)|}{\sqrt{\varepsilon}}\frac{f(c_1+\rho)}{\dot\psi(c_1+\rho)}=O(\sqrt{\varepsilon})$,
- the integral $\int_{c_1+\rho}^{c_2}\log|\psi|\frac{d}{dt}\left(\frac{f}{\dot\psi}\right)dt$ is finite,
- $\lim_{\rho\to0}\int_{c_1}^{c_1+\rho}\log(\varepsilon+(1-\varepsilon)\psi^2)\frac{d}{dt}\left(\frac{f}{\dot\psi}\right)dt=0$,

we obtain

$$\begin{aligned}\frac{1}{\sqrt{\varepsilon}}&\int_{c_1}^{c_2}\frac{f\psi}{\varepsilon+(1-\varepsilon)\psi^2}dt\\&=-\frac{\log(\varepsilon)}{\sqrt{\varepsilon}}\frac{f(c_1+)}{2\dot\psi(c_1+)}+\frac{1}{\sqrt{\varepsilon}}\left(\frac{f(c_2)}{\dot\psi(c_2)}\log|\psi(c_2)|\right.\\&\quad\left.-\int_{c_1}^{c_2}\log|\psi|\frac{d}{dt}\left(\frac{f}{\dot\psi}\right)dt\right)+o(1).\end{aligned}$$

In particular, there are no terms of order zero for such integrals. Similar computations for L_- show that

$$\begin{aligned}\frac{1}{\sqrt{\varepsilon}}&\int_{c_3}^{c_4}\frac{f\psi}{\varepsilon+(1-\varepsilon)\psi^2}dt\\&=\frac{\log(\varepsilon)}{\sqrt{\varepsilon}}\frac{f(c_4-)}{2\dot\psi(c_4-)}+\frac{1}{\sqrt{\varepsilon}}\left(-\frac{f(c_3)}{\dot\psi(c_3)}\log|\psi(c_3)|\right.\\&\quad\left.-\int_{c_3}^{c_4}\log|\psi|\frac{d}{dt}\left(\frac{f}{\dot\psi}\right)dt\right)+o(1).\end{aligned}$$

(c) We need to compute the limit of $\int_{T_3}\frac{\varepsilon\langle D_t^\varepsilon\dot\gamma,I^\varepsilon\dot\gamma\rangle_\varepsilon}{\varepsilon+(1-\varepsilon)\psi^2}dt$ as $\varepsilon\to0$. The integrals of the type in (a) with $f(t)=\frac{\varepsilon\cos\theta}{b_\varepsilon}k_{1,0}$, $f(t)=\psi k_{1,1}dt$, $f(t)=\frac{\sin\theta}{b_\varepsilon+b_0}k_{0,1}$, $f(t)=\frac{\varepsilon\sin\theta}{b_\varepsilon+b_0}k_{1,1}$ will all vanish in the limit, either because the functions vanish on $W^\pm$ or because they are multiplied by ε. Using (b) with $f(t)=k_{0,1}$ has no terms of degree zero. It follows that the only part that can give terms of degree zero in ε is

$$\int_{T_3}\frac{\sqrt{\varepsilon}\cos\theta}{b_\varepsilon(\varepsilon+(1-\varepsilon)\psi^2)}k_{0,0}dt.$$

We see that if we let ε approach zero and then let t move towards $W^\pm$, then the limit is

$$\cos\theta(t\pm)(-a(\gamma(t\pm))\cos\theta(t\pm)\sin\theta(\pm t)+\dot\psi(t\pm)).$$

Outside the cut-locus, we can simplify this expression as $\gamma(t)\notin\mathrm{char}(\Sigma)$, then $\psi(t)=0$ is equivalent to $\sin\theta(t)=0$. Furthermore, since $\psi(t)=0$ for $t\in W^\pm$,

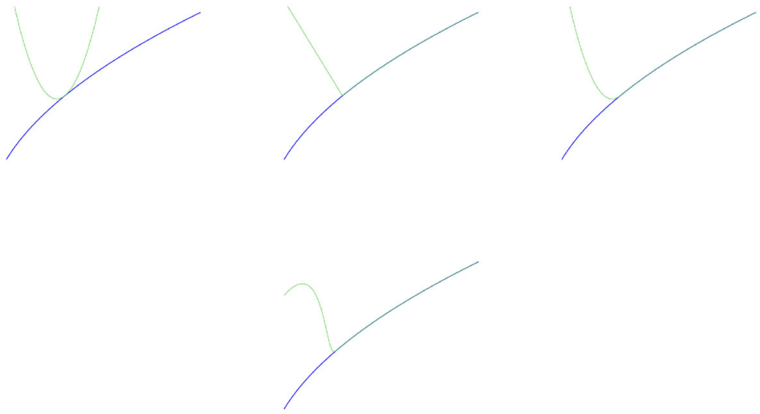

Fig. 2 The figure shows four cases where the boundary $\partial\Sigma$ in green intersects the characteristic foliation tangent to E in blue. The three first are compatible with (B). In the first figure, the boundary is only tangent at an isolated point, and its second derivative does not follow the curve in blue. The second and the third figure represent respectively a C^1- and a C^2-singularity. In the fourth picture, the boundary smoothly becomes tangent to E which is not compatible with (B) (Color figure online)

we have

$$\operatorname{sign}(\dot{\psi}(t\pm)) = \pm \operatorname{sign}(\psi(t\pm)).$$

$\square$

Contributions from corners

Finally, we consider elements of S. We observe that if $v, w \in T\Sigma$ with oriented angle ϕ_j^ε relative to h^ε, then

$$\begin{aligned}\lim_{\varepsilon\to 0} |\phi^\varepsilon| &= \lim_{\varepsilon\to 0} \cos^{-1} \frac{|\langle v, w\rangle_{h^\varepsilon}|}{\|v\|_{h^\varepsilon}\|w\|_{h^\varepsilon}} \\ &= \begin{cases} |\phi_j|, & v, w \in E\cap T\Sigma, \\ \frac{\pi}{2}, & v \in E\cap T\Sigma, w \notin E\cap T\Sigma, \\ \frac{\pi}{2}(1-s), & v, w \notin E\cap T\Sigma, s = \operatorname{sign}(\beta(v)\beta(w)). \end{cases}\end{aligned}$$

The result again follows by writing the Gauss-Bonnet formula with boundary for the g^ε metric and taking the limit $\varepsilon \to 0$. $\square$

Remark 6.3 If (B) does not hold, these we can easily find examples of $\psi(t)$ such that the integral $\int_{c_1}^{c_1+\rho} \frac{\sqrt{\varepsilon}}{\varepsilon+\psi(t)^2}\, dt$ approach ∞ as $\varepsilon\to\infty$, e.g., $\psi(t) = Ct^2$. For finding a Gauss-Bonnet formula in this case, one would need to establish exactly which part of the integral $\int_{c_1}^{c_2} \frac{\sqrt{\varepsilon}\dot{\theta} b_\varepsilon}{\varepsilon+(1-\varepsilon)\psi^2} dt$ has order 0 relative to ε for any general ψ.

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