Holonomy of H-type Foliations

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Abstract. On an H-type manifold we describe the collection of endomorphisms of $\mathcal{H}_p$ induced by Bott-parallel transport around loops at a point p. This is a generalization of the idea of Riemannian holonomy from Riemannian geometry that respects the intrinsic structure of the horizontal bundle; the study of this horizontal holonomy is then related to the structure of the local submersions and the Riemannian holonomy of the base space.

1. Introduction

The notion of considering the group of endomorphisms of the tangent space to a Riemannian manifold generated by parallel transport for some affine connection is now classical; it is well-understood that there is a strong relationship between these holonomy groups and topological information for the manifold. The work of Cartan [**Car26**] classifying symmetric spaces and Berger [**Ber55**] giving the classification of holonomy groups for nonsymmetric spaces (and much work thereafter) has thoroughly established the relationship between these structures.

With motivations arising from foliation theory and sub-Riemannian geometry, the authors introduced the notion of H-type foliations [**BGRVM22**] that are a natural setting in which to study many important step 2 sub-Riemannian manifolds via consideration of the transverse space to a foliation under a limiting process; these have been sucessfully used to establish several desirable results generalizing Riemannian ones as well as to give new methods of understanding earlier results given via an intrinsic approach. For example, in [**BGRVM19**] the authors established sharp Laplacian and Hessian comparison theorems.

In this article we consider a notion of holonomy for H-type foliations determined by an adapted metric connection. Because this connection respects the decomposition of the foliation this induces an horizontal holonomy group consisting of endomorphisms of the horizontal spaces, and since the construction is invariant under the rescaling by penalty metric it can be seen as measuring the holonomy of the sub-Riemannian structure. We give some classification results related to the earlier horizontal Einstein condition in [**BGRVM22**].

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1.1. Riemannian holonomy. On a Riemannian manifold $(\mathbb{M}, g)$ equipped with a metric connection ∇, there is a notion of parallel transport of vector fields; given a curve $\gamma\colon [0,T] \to \mathbb{M}$ and a vector $x \in T_{\gamma(0)}\mathbb{M}$, we say that a vector field $X \in \Gamma(\gamma^* T\mathbb{M})$ such that $X(\gamma(0)) = x$ and $\nabla_{\dot\gamma} X = 0$ is a parallel transport for x along γ. This induces an isomorphism

$$\tau_{t,\gamma}\colon T_{\gamma(0)}\mathbb{M} \to T_{\gamma(t)}\mathbb{M}$$

for all $0 \le t \le T$. In particular, if γ is a loop at a fixed point $\gamma(0) = \gamma(T) = p \in \mathbb{M}$ this gives an endomorphism $\tau_\gamma := \tau_{T,\gamma} \in \mathrm{End}(T_p\mathbb{M})$.

Definition 1.1. We denote by $\mathbf{Hol}(\mathbb{M}, p)$ the family of endomorphisms of $T_p\mathbb{M}$ induced by ∇-parallel transport along loops at p. This forms a group by composition, and is called the holonomy group of $\mathbb{M}$ at p. Restricting to only the loops homotopic to the constant loop at p generates the restricted holonomy group at p, $\mathbf{Hol}^0(\mathbb{M}, p)$.

Proposition 1.2. *Let $(\mathbb{M}, g)$ be a connected, n-dimensional Riemannian manifold with metric connection ∇.*

- $\mathbf{Hol}(\mathbb{M}, p)$ *is independent of the choice of $p \in \mathbb{M}$, and so we simply write* $\mathbf{Hol}(\mathbb{M})$.
- $\mathbf{Hol}(\mathbb{M})$ *is a Lie subgroup of $O(n, \mathbb{R})$.*
- *If $\mathbb{M}$ is orientable then it is a Lie subgroup of $SO(n, \mathbb{R})$.*
- *Every element of* $\mathbf{Hol}(\mathbb{M})$ *is an isometry.*

Proof. The proofs are straightforward, we refer to [**KN96a**, chapter 2]. $\square$

One can characterize the infinitesimal generators of the holonomy group in terms of the associated Riemann curvature tensor.

Theorem 1.3 (Ambrose-Singer [**AS53**]). *Let $(\mathbb{M}, g)$ be a Riemannian manifold with metric connection ∇. For $p \in \mathbb{M}$, the Lie algebra $\mathfrak{hol}(\mathbb{M}, p)$ of $\mathbf{Hol}(\mathbb{M}, p)$ is exactly the sub-algebra of $\mathfrak{so}(T_p\mathbb{M})$ generated by the elements $\tau_\gamma^{-1} \circ R(\tau_\gamma u, \tau_\gamma v) \circ \tau_\gamma$ where u, v run through $T_p\mathbb{M}$ and γ through the set of piecewise-smooth loops based at $p \in \mathbb{M}$.*

Proof. A modern proof is given in [**Bes87**, theorem 10.58] and a note following 10.59. $\square$

Remark 1.4. We emphasize a note made in [**Bes87**], that this result remains true without modification for any metric connection ∇.

We can also simplify the study of holonomy groups to those that are irreducible.

Theorem 1.5 (deRham [**deR52**]). *Let $(\mathbb{M}, g)$ be a complete, simply connected Riemannian manifold. Then $\mathbf{Hol}(\mathbb{M}) = \mathbf{Hol}^0(\mathbb{M})$ is reducible as a direct product if and only if $(\mathbb{M}, g)$ is reducible as a Riemannian product.*

1.2. Holonomy associated to Levi-Civita connection. In particular, the holonomy group $\mathbf{Hol}(\mathbb{M})$ associated to the Levi-Civita connection is referred to as the Riemannian holonomy group; there is a complete classification of the possible Riemannian holonomies, as follows.

Theorem 1.6 (Berger [**Ber55**], Simons [**Sim62**]). *Suppose $(\mathbb{M}, g)$ is a Riemannian manifold with irreducible $\mathbf{Hol}^0(\mathbb{M})$. Then one of the following (nonexclusive) cases occur:*

- $\mathbb{M}$ *is locally symmetric and is of rank at least 2, or*
- $\mathbf{Hol}^0(\mathbb{M})$ *acts transitively on the sphere.*

Simons' proof [**Sim62**] relies on the Ambrose-Singer Theorem 1.3. One assumes that $\mathbf{Hol}^0(\mathbb{M})$ does not act transitively on the sphere, and after an algebraic argument one concludes the proof using the equivalence of local symmetry with the statement $\nabla R = 0$ shown by Cartan. A modern, geometric proof based on the holonomy of submanifolds was established by Olmos [**Olm05**].

The holonomy of symmetric spaces is rather involved, but is completely classified by a series of results due to Cartan [**Car26, Car27**]. See [**KN96b**] for an introduction to the holonomy of symmetric spaces.

In the nonsymmetric case, the list of possible holonomy groups is rather short. This is a consequence of a classification of Lie groups acting transitively on the sphere due to Cartan.

THEOREM 1.7 (Berger [**Ber55**], Simons [**Sim62**]). *For an irreducible, simply connected, nonsymmetric Riemannian manifold manifold* $\mathbb{M}$*, one of the following cases occurs:*

$dim(\mathbb{M})$	$\mathbf{Hol}^0(\mathbb{M})$	*Type*
n	$O(n)$	*Generic*
n	$SO(n)$	*Oriented*
$2m$	$U(m)$	*Kähler*
$2m$	$SU(m)$	*Calabi-Yau*
$4m$	$Sp(m) \cdot Sp(1)$	*Quaternion-Kähler*
$4m$	$Sp(m)$	*Hyperkähler*
7	G_2	G_2 *manifold*
8	$Spin(7)$	$Spin(7)$ *manifold*

This is the complete list of Lie groups acting transitively on the sphere, with the exception of $T \cdot Sp(m)$ and $Spin(9)$. Topological considerations rule out $T \cdot Sp(m)$. $Spin(9)$ occurs, but only for symmetric spaces. The remainder have been shown to exist explicitly; in particular the existence of G_2 and $Spin(7)$ manifolds was difficult, but was established by Bryant in [**Bry87, BS89**]. See [**Bes87**, chapter 10] for a complete discussion.

1.3. H-type Foliations.

We recall here H-type foliations from [**BGRVM22**].

Let $(\mathbb{M}, \mathcal{H}, g)$ be a totally-geodesic foliation with leaves $\mathcal{V}$ and Riemannian bundle-like metric $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$. There is a natural connection well-adapted to this structure.

Definition 1.8. There exists a unique metric connection ∇ on $(\mathbb{M}, \mathcal{H}, g)$ such that

(1) $\mathcal{H}$ and $\mathcal{V}$ are ∇-parallel,
(2) The torsion T of ∇ satisfies
  - $T(\mathcal{H}, \mathcal{H}) \subset \mathcal{V}$,
  - $T(\mathcal{V}, \mathcal{V}) = 0$
(3) For every $X, Y \in \Gamma(\mathcal{H}), Z, V \in \Gamma(\mathcal{V})$,
  - $\langle T(X,Z), Y\rangle_{\mathcal{H}} = \langle T(Y,Z), X\rangle_{\mathcal{H}}$
  - $\langle T(Z,X), V\rangle_{\mathcal{V}} = \langle T(V,X), Z\rangle_{\mathcal{V}}$.

This is called the Hladky-Bott connection. It can be explicitly written in terms of the Levi-Civita connection ∇^g as

$$\nabla_X Y = \begin{cases} \mathrm{pr}_{\mathcal{H}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{H}) \\ \mathrm{pr}_{\mathcal{H}}[X,Y] & Y \in \Gamma(\mathcal{H}), X \in \Gamma(\mathcal{V}) \\ \mathrm{pr}_{\mathcal{V}}[X,Y] & Y \in \Gamma(\mathcal{V}), X \in \Gamma(\mathcal{H}) \\ \mathrm{pr}_{\mathcal{V}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{V}) \end{cases}$$

and has torsion

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = -\mathrm{pr}_{\mathcal{V}}[\mathrm{pr}_{\mathcal{H}} X, \mathrm{pr}_{\mathcal{H}} Y].$$

For each $Z \in \Gamma(T\mathbb{M})$ we define an endomorphism $J_Z \in \mathrm{End}(\Gamma(T\mathbb{M}))$ dual to the Bott torsion by

$$g_{\mathcal{H}}(J_Z X, Y) = g_V(Z, T(X,Y))$$

for all $X, Y, Z \in \Gamma(T\mathbb{M})$.

Definition 1.9. We say that the map J_Z satisfies the H-type condition if for every $Z \in \Gamma(\mathcal{V})$ is an isometry:

$$g(J_Z X, J_Z Y) = \|Z\|^2 g(X,Y)$$

for all $X, Y \in \Gamma(T\mathbb{M})$.

We then call $(\mathbb{M}, \mathcal{H}, g)$ an H-type foliation.

Remark 1.10. These objects were introduced in [**BGRVM22**] as a generalization of Kaplan's H-type groups [**Kap83**] to curved spaces and are of interest in sub-Riemannian geometry; by defining a family of penalty metrics [**Mon02**]

$$g_\varepsilon = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon} g_{\mathcal{V}}$$

one can show that there is convergence as $\varepsilon \to 0^+$ to the sub-Riemannian manifold $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ in the Gromov-Hausdorff sense, that the Riemannian distance functions converge to the sub-Riemannian Carnot-Caratheodory metric [**BGRVM19**], and that it is possible to recover a number of sub-Riemannian results from appropriate Riemannian analogues in this limit. One must be very careful as the standard notions of curvature (associated to the Levi-Civita connection, for example) fail spectacularly in the limit.

We begin by classifying these manifolds by the behaviour of the covariant derivative of the torsion.

Definition 1.11. Let $(M, \mathcal{H}, g)$ be an H-type foliation.

(1) If $\delta_{\mathcal{H}} T = 0$ we say $(\mathbb{M}, \mathcal{H}, g)$ is Yang-Mills.
(2) If $\nabla_{\mathcal{H}} T = 0$ we say $(\mathbb{M}, \mathcal{H}, g)$ has horizontally parallel torsion.
(3) If $\nabla T = 0$ we say $(\mathbb{M}, \mathcal{H}, g)$ has (completely) parallel torsion.

We will need a few results which we collect here.

Proposition 1.12. *H-type foliations are Yang-Mills.*

Proposition 1.13. *Let $(\mathbb{M}, \mathcal{H}, g)$ have horizontally parallel torsion. Then*

(1) $(\nabla_X J)Y = -(\nabla_Y J)X$*, and*
(2) $R(X,Y)Z = R_{\mathcal{H}}(X,Y)Z + R_{\mathcal{V}}(X,Y)Z + (\nabla_Z T)(X,Y)$

for all $X, Y, Z \in \Gamma(T\mathbb{M})$*, where we have defined*

$$R_{\mathcal{H}}(X,Y)Z = R(\mathrm{pr}_{\mathcal{H}}X, \mathrm{pr}_{\mathcal{H}}Y)\mathrm{pr}_{\mathcal{H}}Z$$
$$R_{\mathcal{V}}(X,Y)Z = R(\mathrm{pr}_{\mathcal{V}}X, \mathrm{pr}_{\mathcal{V}}Y)\mathrm{pr}_{\mathcal{V}}Z.$$

Proof. We refer to [**BGRVM22**]. □

2. Adapted holonomy of foliations

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation, and denote by ∇ the Bott connection. If $p \in \mathbb{M}$ and $\gamma\colon [0,T] \to \mathbb{M}$ is a piecewise-smooth curve, let us denote by $\tau_{t,\gamma}\colon T_p\mathbb{M} \to T_{\gamma(t)}\mathbb{M}$ the ∇-parallel transport along γ. Since ∇ is metric, $\tau_{t,\gamma}$ is an element of the orthogonal group $O(T_p\mathbb{M}, T_{\gamma(t)}\mathbb{M})$. Moreover, since ∇ preserves $\mathcal{H}$ and $\mathcal{V}$ we can introduce holonomy groups for the horizontal and vertical spaces.

Definition 2.1. Denote by $\mathcal{C}_p$ the set of piecewise-smooth loops based at $p \in \mathbb{M}$. We call the subgroup of $O(\mathcal{H}_p)$ generated by the set of all $\tau_\gamma|_{\mathcal{H}}$ $(\gamma \in \mathcal{C}_p)$ the horizontal holonomy group at p denoted by $\mathbf{Hol}(\mathcal{H}, p)$. We take analogous definitions for $O(\mathcal{V}_p)$, $\tau_\gamma|_{\mathcal{V}}$, and $\mathbf{Hol}(\mathcal{V}, p)$.

Restricting to the subset $\mathcal{C}_p^0 \subseteq \mathcal{C}_p$ consisting only of loops homotopic to the identity, we get the restricted holonomy subgroups denoted by $\mathbf{Hol}^0(\mathcal{H}, p), \mathbf{Hol}^0(\mathcal{V}, p)$.

The horizontal holonomy groups at each point are all isomorphic. This enables us to talk generically of the horizontal holonomy groups of $(\mathbb{M}, \mathcal{H}, g)$ which we will denote $\mathbf{Hol}(\mathcal{H}), \mathbf{Hol}^0(\mathcal{H})$. Similarly, we can talk of $\mathbf{Hol}(\mathcal{V}), \mathbf{Hol}^0(\mathcal{V})$.

Remark 2.2. This entire construction is independent of the sub-Riemannian penalty metric $g_\varepsilon = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon} g_{\mathcal{V}}$. As a consequence one can interpret the horizontal holonomy group as a sub-Riemannian notion of holonomy.

We have the following theorem describing the infinitesimal generators of the holonomy groups.

Theorem 2.3. *Let* $(\mathbb{M}, \mathcal{H}, g)$ *be an H-type foliation with horizontally parallel torsion. For* $p \in \mathbb{M}$*, the Lie algebra* $\mathfrak{hol}(\mathcal{H}, p)$ *of* $\mathbf{Hol}(\mathcal{H}, p)$ *is exactly the sub-algebra of* $\mathfrak{so}(\mathcal{H}_p)$ *generated by the elements* $\tau_\gamma^{-1} \circ R_{\mathcal{H}}(\tau_\gamma u, \tau_\gamma v) \circ \tau_\gamma$ *where* u, v *run through* $\mathcal{H}_p$ *and* γ *through* $\mathcal{C}_p$*.*

Proof. This is a straightforward consequence of the Riemannian Ambrose-Singer Theorem 1.3 and Proposition 1.13. □

Remark 2.4. Analogously to Remark 1.4, the theorem will remain true for any connection with horizontally parallel torsion.

Corollary 2.5. *Suppose* $(\mathbb{M}, \mathcal{H}, g)$ *is an H-type foliation with completely parallel torsion. If* $m \geq 4$ *then* $R_{\mathcal{H}} = 0$ *and so* $\mathbf{Hol}(\mathcal{H}) = \mathrm{Id}$.

3. Horizontal holonomy of H-type foliations

3.1. H-type submersions. To begin, we will consider H-type submersions; these are a subclass of H-type foliations that have a simple global structure.

Definition 3.1. We say that $(\mathbb{M}, \mathcal{H}, g, \pi, \mathbb{B}, j)$ is an H-type submersion if $(\mathbb{M}, \mathcal{H}, g)$ is an H-type foliation with horizontally parallel torsion and $\pi\colon (\mathbb{M}, g) \to (\mathbb{B}, j)$ is a Riemannian submersion with horizontal space $\mathcal{H}$. We call $\mathbb{B}$ the base space. We write $\nabla^{\mathbb{M}}, \nabla^{\mathbb{B}}$ for the Levi-Civita connections on $\mathbb{M}$ and $\mathbb{B}$, and ∇ for the Bott connection.

We will characterize the holonomy of H-type submersions via the Riemannian holonomy of the base space. To begin, we will leverage the fact that we are working on global submersions by defining the notion of basic vector field that will be key to our study of the holonomy.

Definition 3.2. We say $X \in \Gamma(T\mathbb{M})$ is projectable if there exists $\bar{X} \in \Gamma(T\mathbb{B})$ such that $d\pi(X) = \bar{X}$. We will say X and $\bar{X}$ are π-related. Moreover, if $X \in \Gamma(\mathcal{H})$ is projectable we say X is basic.

Basic vector fields are ubiquitous in the study of submersions and foliations. See for example [**Bes87, Ton88, Mol88, Her60**].

Lemma 3.3. *Associated to any vector field $\bar{X} \in \Gamma(T\mathbb{B})$ there is an unique basic π-related vector field $X \in \Gamma(\mathcal{H})$.*

Lemma 3.4. *Let $X, Y \in \Gamma(T\mathbb{M})$ be projectable, and let $Z \in \Gamma(\mathcal{V})$.*
- *$[X, Y]$ is projectable and π-related to $[\bar{X}, \bar{Y}]$.*
- *$[X, Z]$ is vertical.*

Proof. We refer the reader to [**Bes87**] for standard proofs of the lemmas. □

Let $R^{\mathbb{B}}$ denote the Riemann curvature tensor on $\mathbb{B}$ for the Levi-Civita connection, and $R_{\mathcal{H}}$ be as in Proposition 1.13. Our strategy is to relate these tensors via basic vector fields, then apply our Ambrose-Singer type Theorem 2.3 to determine the horizontal holonomy group of $\mathbb{M}$. We begin with the following lemma.

Lemma 3.5. *For basic vector fields X, Y on $\mathbb{M}$,*
- *$\nabla_X Y$ is basic and π-related to $\nabla^{\mathbb{B}}_{\bar{X}} \bar{Y}$*
- *$\mathrm{pr}_{\mathcal{H}}[X, Y]$ is basic and π-related to $[\bar{X}, \bar{Y}]$*

Proof. First, that $\nabla^{\mathbb{B}}_{\bar{X}} \bar{Y}$ is π-related to $\mathrm{pr}_{\mathcal{H}}(\nabla^{\mathbb{M}}_X Y)$ follows from writing the lift of $\nabla^{\mathbb{B}}_{\bar{X}} \bar{Y}$ in a local coordinate chart using the Christoffel symbols. Then since X and Y are horizontal, $\nabla_X Y = \mathrm{pr}_{\mathcal{H}}(\nabla^{\mathbb{M}}_X Y)$ follows from the properties of the Bott connection and the first claim is established.

The second claim follows directly from Lemma 3.4. □

Now we can see that $R_{\mathcal{H}}$ is the appropriate curvature tensor to relate to $R^{\mathbb{B}}$ via π-related vector fields.

Lemma 3.6. *Let X, Y, Z be basic vector fields on $\mathbb{M}$, and denote by $R^{\mathbb{B}}$ the Riemann curvature tensor on $\mathbb{B}$. Then $R_{\mathcal{H}}(X, Y)Z$ is basic and is π-related to $R^{\mathbb{B}}(\bar{X}, \bar{Y})\bar{Z}$.*

Proof. Observe that on $\mathbb{M}$,

$$\nabla_{\mathrm{pr}_{\mathcal{V}}[X,Y]} Z = -\nabla_{T(X,Y)} Z = -\mathrm{pr}_{\mathcal{H}}[T(X, Y), Z] = 0$$

where we use properties of the Bott connection for the first two equalities and Lemma 3.4 for the last. As a consequence,

$$R_{\mathcal{H}}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\mathrm{pr}_{\mathcal{H}}[X,Y]} Z$$

Applying the previous lemma, we compute on $\mathbb{B}$ that

$$\begin{aligned} R^{\mathbb{B}}(\bar{X}, \bar{Y})\bar{Z} &= \nabla^{\mathbb{B}}_{\bar{X}} \nabla^{\mathbb{B}}_{\bar{Y}} \bar{Z} - \nabla^{\mathbb{B}}_{\bar{Y}} \nabla^{\mathbb{B}}_{\bar{X}} \bar{Z} - \nabla^{\mathbb{B}}_{[\bar{X}, \bar{Y}]} \bar{Z} \\ &= \overline{\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\mathrm{pr}_{\mathcal{H}}[X,Y]} Z}. \end{aligned}$$

and the lemma follows. □

Denote by $\overline{\mathbf{Hol}}(\mathbb{B},\bar{p})$ the holonomy group of $\mathbb{B}$ at $\bar{p} \in \mathbb{B}$ for the Levi-Civita connection, and by $\overline{\mathfrak{hol}}(\mathbb{B},\bar{p})$ its Lie algebra. These are again isomorphic for all $\bar{p} \in \mathbb{B}$, and so we can write $\overline{\mathbf{Hol}}(\mathbb{B})$ and $\overline{\mathfrak{hol}}(\mathbb{B})$.

Notation 3.7. *In the following, we will consistently denote by $\gamma\colon [0,T] \to \mathbb{M}$ piecewise-smooth curves, set $p = \gamma(0)$, and their ∇-parallel transport by $\tau_{t,\gamma}\colon T_p\mathbb{M} \to T_{\gamma(t)}\mathbb{M}$. Frequently we write $\tau_\gamma = \tau_{T,\gamma}$ when clear from context, particularly when $\gamma(T) = \gamma(0) = p$ so $\gamma \in \mathcal{C}_p$. We will also use overbars to indicate the analogous concepts on the base space $\mathbb{B}$.*

THEOREM 3.8. *For an H-type submersion $(\mathbb{M},\mathcal{H},g,\pi,\mathbb{B})$,*

$$\mathbf{Hol}^0(\mathcal{H}) \cong \overline{\mathbf{Hol}}^0(\mathbb{B})$$

PROOF. For $p \in \mathbb{M}$, we want to relate the Lie algebras $\mathfrak{hol}(\mathcal{H},p)$ and $\overline{\mathfrak{hol}}(\mathbb{B},\pi(p))$. Using the Riemannian Ambrose-Singer theorem and our Theorem 2.3, this is reduced to studying the parallel transports $\tau_\gamma, \bar{\tau}_{\bar{\gamma}}$ and curvature tensors $R_\mathcal{H}$ and $R^\mathbb{B}$.

We need the following

Lemma 3.9. *Suppose γ is a piecewise-smooth curve on $\mathbb{M}$ and let $\bar{\gamma}$ be the piecewise-smooth curve on $\mathbb{B}$ determined by $\bar{\gamma}(t) = \pi(\gamma(t))$. Let $u \in \mathcal{H}_p$ and set $\bar{u} = d_p\pi(u) \in T_{\bar{p}}\mathbb{B}$. Then for all $t \in [0,T]$,*

$$d_{\gamma(t)}\pi(\tau_{t,\gamma}(u)) = \bar{\tau}_{t,\bar{\gamma}}(\bar{u}).$$

PROOF. Let $Y(t) = \tau_{t,\gamma}(u)$ and $\bar{Y}\colon [0,T] \to T\mathbb{B}$ be the pushforward

$$\bar{Y}(t) = d_{\gamma(t)}\pi(Y(t)) \in T_{\bar{\gamma}(t)}\mathbb{B}.$$

Then

$$D_t^\mathbb{B}\bar{Y} = d_{\gamma(t)}\pi(D_tY) = 0$$

since Y is the ∇-parallel transport of u (where $D^\mathbb{B}, D^\mathbb{M}, D$ are the respective covariant derivatives along γ, which do not depend on the choice of extension of $Y, \bar{Y}, \gamma'$, and $\bar{\gamma}'$). Thus $\bar{Y}$ is the $\nabla^\mathbb{B}$-parallel transport of $\bar{u}$ along $\bar{\gamma}$, and the lemma follows from the uniqueness of parallel transport. □

Fix $p \in \mathbb{M}$; let $\gamma \in \mathcal{C}_p$ and $\bar{\gamma} = \pi\gamma \in \bar{\mathcal{C}_p}$, $x, y, z \in \mathcal{H}_p$ and π-related vectors $\bar{x}, \bar{y}, \bar{z} \in T_{\pi(p)}\mathbb{B}$. Applying Lemma 3.6 and Lemma 3.9 gives us that

$$d_p\pi(\tau_\gamma^{-1}R_\mathcal{H}(\tau_\gamma(x),\tau_\gamma(y))\tau_\gamma z) = \bar{\tau}_{\bar{\gamma}}^{-1}R^\mathbb{B}(\bar{\tau}_{\bar{\gamma}}(\bar{x}),\bar{\tau}_{\bar{\gamma}}(\bar{y}))\bar{\tau}_{\bar{\gamma}}\bar{z}$$

and so by Theorem 2.3

$$\mathfrak{hol}(\mathcal{H},p) \cong \overline{\mathfrak{hol}}(\mathbb{B},\pi(p))$$

from which the theorem follows. □

Corollary 3.10. *If $(\mathbb{M},\mathcal{H},g)$ is a simply-connected H-type submersion with base $(\mathbb{B},j)$ then $\mathbf{Hol}(\mathcal{H}) \cong \overline{\mathbf{Hol}}(\mathbb{B})$.*

There is a complete classification of H-type submersions in [**BGRVM22**], and so we have a complete list of the possible horizontal holonomies of H-type submersions.

3.2. Holonomy of H-type foliations with horizontally parallel Clifford structure. We now consider the more general setting of foliations that are not globally submersions. In view of Theorem 2.3, to study the horizontal holonomy group $\mathbf{Hol}(\mathcal{H})$ the study of the symmetries of the horizontal endomorphisms $R_{\mathcal{H}}$ is of paramount importance. We recall now the notion of horizontally parallel Clifford structures that naturally arise on H-type foliations.

Definition 3.11. Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with horizontally parallel torsion. We say that it has parallel horizontal Clifford structure if there exists a smooth bundle map $\Psi \colon \mathcal{V} \times \mathcal{V} \to \mathrm{Cl}_2(\mathcal{V})$ such that for every $Z_1, Z_2 \in \Gamma(V)$,

$$(\nabla_{Z_1} J)_{Z_2} = J_{\Psi(Z_1, Z_2)}$$

where we have naturally extended J to $\mathrm{Cl}_2(\mathcal{V})$ as $J_{Z \cdot W} = J_Z J_W$.

There is a rigidity result as follows.

Lemma 3.12. *Suppose Ψ is a horizontally parallel Clifford structure. Then there exists a $\kappa \in \mathbb{R}$ such that*

$$\Psi(u, v) = -\kappa(u \cdot v + \langle u, v \rangle)$$

and the sectional curvature of the leaves tangent to $\mathcal{V}$ is κ^2. If moreover the torsion is completely parallel then $\kappa = 0$ and the leaves are flat.

Given an horizontally parallel Clifford structure, we recall a useful lemma for the curvature tensors $R_{\mathcal{H}}$.

Lemma 3.13. *Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with $m = \mathbf{rank}(\mathcal{V}) \geq 2$ and parallel horizontal Clifford structure $\Psi(u, v) = -\kappa(u \cdot v + \langle u, v \rangle)$. Let $Z_i, 1 \leq i \leq m$ be a local orthonormal basis for $\mathcal{V}$, and write $J_i = J_{Z_i}, J_{ij} = J_i J_j$. Then*

$$[R_{\mathcal{H}}(u, v), J_i] = \kappa \sum_{j=1, j \neq i}^{m} \Big(\langle J_j u, v \rangle J_{ij} - \langle J_{ij} u, v \rangle J_j \Big).$$

for all $u, v \in \mathcal{H}_p, z \in \mathcal{V}_p$.

In particular, we emphasize the following corollary.

Corollary 3.14. *Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with parallel horizontal Clifford structure.*

- *If $\kappa = 0$ then $R_{\mathcal{H}}(u, v)$ commutes with J_z.*
- *If $\mathbb{M}$ is quaternionic, that is if $m = 3$ and $\mathfrak{a}(p) = \{J_z, z \in \mathcal{V}\} \cong \mathfrak{so}(3)$, then $R_{\mathcal{H}}(u, v)$ will preserve $\mathfrak{a}(p)$.*

For proofs, existence results, a classification, and further discussion of horizontally parallel Clifford structures we refer to [**BGRVM22**] and the underlying classification [**MS11**].

We can now accomplish the following structural theorem.

Theorem 3.15. *Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation with parallel horizontal Clifford structure, and write $n = \mathbf{rank}(\mathcal{H}), m = \mathbf{rank}(\mathcal{V})$.*

(a) *If $m = 1$, then $\mathbf{Hol}^0(\mathcal{H})$ is isomorphic to a subgroup of $\mathbf{U}(n/2)$.*
(b) *If $m \geq 2$ and $\kappa = 0$, then $\mathbf{Hol}^0(\mathcal{H})$ is isomorphic to a subgroup of $\mathbf{Sp}(n/4)$.*
(c) *If $\mathbb{M}$ is quaternionic, then $\mathbf{Hol}^0(\mathcal{H})$ is isomorphic to a subgroup of $\mathbf{Sp}(1)\mathbf{Sp}(n/4)$*

PROOF. Let $p \in \mathbb{M}$ be a chosen base point, $\gamma \in \mathcal{C}_p^0$. The strategy of the proof is to explicitly construct appropriate complex and symplectic forms.

(a) If $m = 1$, then $\mathbf{Hol}(\mathcal{V}) = \mathrm{Id}$, so $\tau_\gamma z = z$ for any $z \in \mathcal{V}_p$ and since J is also parallel we have $\tau_\gamma J_z = J_z \tau_\gamma$. Fix a unit vector $z \in \mathcal{V}_p$, and then $J = J_z = T^\flat(\cdot, \cdot, z)^{\sharp_1}$ will be a complex structure on $T_p\mathbb{M}$. We can now define a complex inner product on $\mathcal{H}_p$ by

$$\langle u, w\rangle_{p,\mathbb{C}} = g_{p,\mathcal{H}}(u, w + Jw)$$

We will then have $\langle \tau_\gamma u, \tau_\gamma w\rangle_{p,\mathbb{C}} = \langle u, w\rangle_{p,\mathbb{C}}$, which implies $\tau_\gamma \in \mathbf{U}(n/2)$.

Since $\mathbf{rank}(\mathcal{V}) = 1$ it must be that $R_\mathcal{H}(u, v)$ will commute with J. The result follows from Theorem 2.3.

(b) First, since $\kappa = 0$, we have that J is parallel and that $R_\mathcal{V} = 0$, so $\mathbf{Hol}(\mathcal{V}) = \mathrm{Id}$. Fixing orthogonal unit vectors $z_1, z_2 \in \mathcal{V}_p$ we can define $J_k = J_{z_k}$ for $k = 1, 2$. We note that $J_1 J_2 = -J_2 J_1$ and that $\tau_\gamma J_k = J_k \tau_\gamma$. We use J_1 as a complex structure on $\mathcal{H}_p$ and repeat the argument in (a) to show that, with respect to the appropriate choice of basis, $\tau_\gamma \in \mathbf{U}(n/2)$. Furthermore,

$$\omega(u, v) = \langle J_1 u, v\rangle_\mathcal{H} + i\langle J_1 J_2 u, v\rangle_\mathcal{H}, \qquad u, v \in \mathcal{H}_p,$$

is a complex symplectic form on $\mathcal{H}_p$, meaning that relative to the same basis $\tau_\gamma \in \mathbf{Sp}(n/4)$.

Since $\kappa = 0$ we have that $R_\mathcal{H}(u, v)$ will commute with any J_k by Corollary 3.14. The result follows from Theorem 2.3.

(c) From our assumption, it follows that the subbundle $\{J_z \in \mathbf{End}(\mathcal{H}_p) \colon z \in \mathcal{V}\}$ is preserved under parallel transport. Hence there is some map $Q \in \mathbf{End}(\mathcal{V}_p)$ such that $\tau_\gamma J_z v = J_{Qz}\tau_\gamma v$. Furthermore, we have that for any unit vector $v \in \mathcal{H}_p$,

$$g_{p,\mathcal{V}}(z_1, z_2) = g_{p,\mathcal{H}}(\tau_\gamma J_{z_1} v, \tau_\gamma J_{z_2} v) = g_{p,\mathcal{H}}(J_{Qz_1}\tau_\gamma v, J_{Qz_2}\tau_\gamma v) = g_{p,\mathcal{V}}(Qz_1, Qz_2)$$

so Q is an (orientation preserving) isometry as well. Since $\mathfrak{a}(p)$ is a subalgebra of the skew-symmetric endomorphisms isomorphic to $\mathfrak{sp}(1)$, there exists a unique element $A \in \exp\{J_z \colon z \in \mathcal{V}_p\}$ such that

$$AJ_z A^{-1} z = \tau_\gamma J_z \tau_\gamma^{-1} = J_{Qz}.$$

It follows that $\tilde{\tau_\gamma} := A^{-1}\tau_\gamma$ satisfies $\tilde{\tau_\gamma} J_z = J_z \tilde{\tau_\gamma}$. By similar arguments as in (b), it follows that $\tilde{\tau_\gamma}$ can be made unitary and symplectic.

By Corollary 3.14, $R_\mathcal{H}(u, v)$ will preserve $\mathfrak{a}(p)$. The result follows from Theorem 2.3.

□

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